On dynamics of graph maps with zero topological entropy

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Happy birthday to Prof. Misiurewicz!

Figure: Phoenix flowers (Delonix regia) in Shantou University

Motivation

Theorem (Llibre-Misiurewicz 1993)

If a continuous map f of a graph into itself has positive topological entropy, then there exist sequences $(k_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$ of positive integers such that for each n the map f^{k_n} has an s_n-horseshoe and

$$\lim_{n\to\infty} \frac{1}{k_n} \log s_n = h(f)$$

Llibre, J.; Misiurewicz, M. Horseshoes, entropy and periods for graph maps. Topology 32 (1993), no. 3, 649–664. MR1231969

Motivation

Theorem (Li-Yorke 1975)

Let $f\colon [0,1] \to [0,1]$ *be a continuous map. If* f *has a periodic point with period* 3*, then* f *is Li-Yorke chaotic.*

Observation

Horseshoe implies Li-Yorke chaos.

Corollary

If a continuous map f *of a graph into itself has positive topological entropy, then* f *is Li-Yorke chaotic.*

Li, T.-Y.; Yorke, James A. Period three implies chaos. Amer. Math. Monthly 82 (1975), no. 10, 985–992. MR0385028

Motivation

Theorem (Xiong 1986, Smital 1986)

There exists a continuous map $f\colon [0,1] \to [0,1]$ such that the topological entropy of f is 0, but f is Li-Yorke chaotic.

Theorem (Smital 1986)

Let f be an interval map with zero topological entropy. Then f is Li-Yorke chaotic if and only if f has a non-separable pair.

- Xiong, J. A chaotic map with topological entropy. Acta Math. Sci. (English Ed.) 6 (1986), no. 4, 439–443. MR0924033
- Smítal, J. Chaotic functions with zero topological entropy. Trans. Amer. Math. Soc. 297 (1986), no. 1, 269–282. MR0849479

Motivation

Theorem (Kuchta-Smital 1989)

Let $f\colon [0,1] \to [0,1]$ be a continuous map. One scrambled pair implies Li-Yorke chaos.

Theorem (Franzova-Smital 1991)

Let $f\colon [0,1] \to [0,1]$ be a continuous map. Then f is Li-Yorke chaotic if and only if f has positive topological sequence entropy.

- Kuchta, M.; Smítal, J. Two-point scrambled set implies chaos. European Conference on Iteration Theory (Caldes de Malavella, 1987), 427–430, World Sci. Publ., Teaneck, NJ, 1989. MR1085314
- Franzová, N.; Smítal, J. Positive sequence topological entropy characterizes chaotic maps. Proc. Amer. Math. Soc. 112 (1991), no. 4, 1083–1086. MR1062387

Motivation

Theorem (Kuchta 1990)

For a circle map, one scrambled pair implies Li-Yorke chaos.

Theorem (Hric 2000)

For a circle map, it is Li-Yorke chaotic if and only if f has positive topological sequence entropy.

- Kuchta, M. Characterization of chaos for continuous maps of the circle. Comment. Math. Univ. Carolin. 31 (1990), no. 2, 383–390. MR1077909
- Hric, R. Topological sequence entropy for maps of the circle. Comment. Math. Univ. Carolin. 41 (2000), no. 1, 53–59. MR1756926

Motivation

Theorem (Glasner-Ye 2009)

For an interval map, it has positive topological sequence entropy if and only if it is not tame.

Theorem (Ruette-Snoha 2014)

For a graph map, one scrambled pair implies Li-Yorke chaos.

- Glasner, E; Ye, X. Local entropy theory. Ergodic Theory Dynam. Systems 29 (2009), no. 2, 321–356. MR2486773
- Ruette, S; Snoha, L. For graph maps, one scrambled pair implies Li-Yorke chaos. Proc. Amer. Math. Soc. 142 (2014), no. 6, 2087–2100. MR3182027

Main theorem

Theorem (Li-Oprocha-Y.-Zeng 2017)

Let $f : G \to G$ be a graph map. The following conditions are equivalent:

1. f *is Li-Yorke chaotic;*
2. f *has positive topological sequence entropy;*
3. f *is not tame.*

Li, J; Oprocha, P; Yang, Y; Zeng,T. On dynamics of graph maps with zero topological entropy. Nonlinearity 30 (2017), no. 12, 4260–4276. MR3734136

The idea of the proof

1. Local entropy theory (IN-pair, IT-pair)
2. Results on interval maps
3. The structure of ω-limit set of graph maps

IN tuple and IT-tuple

Definition

Let (X, f) be a topological dynamical system. For a tuple $(A_1, \ldots, A_k)$ of subsets of X, we say that a subset $J \subset \mathbb{N}_0$ is an independence set for $(A_1, \ldots, A_k)$ if for any non-empty fnite subset $I \subset J$, we have $\bigcap_{i \in I} f^{-i} A_{s(i)} \neq \emptyset$ for all $s \in \{1, \ldots, k\}^I$.

Definition

A tuple $(x_1, \ldots, x_k) \in X^k$ is called an IT-tuple (resp. IN-tuple) if for every product neighborhood $U_1 \times \cdots \times U_k$ of $(x_1, \ldots, x_k)$, the tuple $(U_1, \ldots, U_k)$ has an infinite independence set (resp. has arbitrarily long finite independence sets).

IN tuple and IT-tuple

Theorem (Kerr-Li 2007)

Let (X, f) be a topological system. Then

1. *f has positive topological sequence entropy if and only if there exists a non-diagonal IN-pair;*
2. *f is not tame if and only if there exists a non-diagonal IT-pair.*

Kerr, D; Li, H. Independence in topological and C^*-dynamics. Math. Ann. 338 (2007), no. 4, 869–926. MR2317754

Interval result

Theorem (Li, 2011)

Let $f: [0,1] \to [0,1]$ *with* $h(f) = 0$ *and* $x < y \in [0,1]$*. Then the following conditions are equivalent:*

1. (x, y) *is a non-separable pair;*
2. (x, y) *is an IT-pair;*
3. (x, y) *is an IN-pair.*

Li, J. Chaos and entropy for interval maps. J. Dynam. Differential Equations 23 (2011), no. 2, 333–352. MR2802890

- Blokh, A. M. On transitive mappings of one-dimensional branched manifolds. (Russian) Differential-difference equations and problems of mathematical physics (Russian), 3–9, 131, Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev, 1984. MR0884346
- Blokh, A. M. Dynamical systems on one-dimensional branched manifolds. I. (Russian) ; translated from Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. No. 46 (1986), 8–18 J. Soviet Math. 48 (1990), no. 5, 500–508 MR0865783
- Blokh, A. M. Dynamical systems on one-dimensional branched manifolds. II. (Russian) ; translated from Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. No. 47 (1987), 67–77 J. Soviet Math. 48 (1990), no. 6, 668–674 MR0916445
- Blokh, A. M. Dynamical systems on one-dimensional branched manifolds. III. (Russian) ; translated from Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. No. 48 (1987), 32–46 J. Soviet Math. 49 (1990), no. 2, 875–883 MR0916457

Definition

An infinite ω-limit set $\omega_f(x)$ of a graph map is called a solenoid if there exists a sequence of cycles of graphs $(X_n)_{n\in N}$ with periods $(k_n)_{n\in N}$ such that

1. $(k_n)_{n\in N}$ is strictly increasing, $k_1 \geq 1$ and k_{n+1} is the multiple of k_n for all $n \geq 1$;
2. for all $n \geq 1$, $X_{n+1} \subset X_n$, and every connected components of X_n contains the same numbers of the connected components of X_{n+1};
3. $\omega(x, f) \subset \bigcap_{n\geq 1} X_n$ and $\omega(x, f)$ contains no periodic points.

Structure of ω-limit sets of graph maps

Definition

Let X be a finite union of subgraphs of G such that $f(X) \subset X$. We define

$$E(X, f) = \{y \in X : \text{for any neighborhood } U \text{ of } y \text{ in } X, \overline{\mathrm{Orb}_f(U)} = X\}.$$

Definition

Assume that $\omega_f(x)$ is an infinite set but not a solenoid. Let X be the minimal (in the sense of inclusion) cycle of graphs containing $\omega_f(x)$ and denote $E = E(X, f)$. We say that E is a basic set if X contains a periodic point, and circumferential set otherwise.

Sketch of the proof

1. Finite ω-limit set, asymptotic periodic.
2. Solenoid, transfer to the interval case.
3. Existence of a basic set implies positive entropy.
4. Circumferential set, there is no scrambled pair, and the system is tame and null.

Circle map

Question [Li-Oprocha-Y.-Zeng 2017]

Let $f: G \to G$ be a graph map with zero topological entropy. Is it true that: (x, y) is an IN-pair if and only if (x, y) is an IT-pair?

Theorem (Y. 2019)

Let $f: \mathbb{S} \to \mathbb{S}$ with $h(f) = 0$ and $x < y \in \mathbb{S}$. Then the following conditions are equivalent:

1. (x, y) *is a non-separable pair;*
2. (x, y) *is an IT-pair;*
3. (x, y) *is an IN-pair.*

Yang Y. Some properties of circle maps with zero topological entropy, 2019, preprint

Key Lemma

Lemma (Mai 1997)

Let $f : \mathbb{S} \to \mathbb{S}$ *be a circle map with zero topological entropy. Assume that* $\mathrm{Fix}(f) \neq \emptyset$*. Then there exists a lifting* F *of* f *and an closed subinterval* $I = [a, b]$ *of* $\mathbb{R}$ *with* $1 \leq b - a < 2$ *such that* $F(I) \subset I$*.*

Mai, J. Some dynamical system properties of self-maps of the circle and conditions equivalent to them. Adv. in Math. (China) 26 (1997).MR1479019

Sketch of the proof

- f has periodic points (For example)
  $\langle x, y\rangle$ is a non-separable pair of $f \xrightarrow{e(x')=x,e(y')=y} \langle x', y'\rangle$ is a non-separable pair of $F|_I \xrightarrow{(Li-2011)} \langle x', y'\rangle$ is a IN-pair of $F|_I$ $\xrightarrow{e(x')=x,e(y')=y} \langle x, y\rangle$ is a IN-pair of f.
- f has no periodic points
  There is no non-separable pairs, no IN-pairs and no IT-pairs in the system.

Background

Definition

A dynamical system (X, f) is mean equicontinuous or mean Lyapunov stable if for every $\epsilon > 0$ there is $\delta > 0$ with the property that for every two points $x, y \in X$, condition $d(x, y) < \delta$ implies

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} d(f^k(x), f^k(y)) < \epsilon$$

Theorem (Li-Tu-Ye 2015)

If a dynamical system (X, f) is mean equicontinuous, then every ergodic invariant measure has discrete spectrum.

Li, J.; Tu, S.; Ye, X. Mean equicontinuity and mean sensitivity. Ergodic Theory Dynam. Systems 35 (2015), no. 8, 2587–2612. MR3456608.

Main theorems

Theorem (Li-Oprcha-Y.-Zeng 2017)

Let $f: G \to G$ be a graph map. Then f has zero topological entropy if and only if it is locally mean equicontinuous, that is for every $x \in G$, $(\overline{Orb_f(x, f)}, f)$is mean equicontinuous.

Sketch of the proof

Case 1: $\omega_f(x)$ is finite

Let (X, f) be a topological dynamical system and let $x \in X$. If $\omega_f(x)$ is finite, then $(\overline{Orb(x, f)}, f)$ is equicontinuous.

Case 2: $\omega_f(x)$ is infinite and is a solenoid

Let $f : G \to G$ be a graph map and $x \in G$. If $\omega_f(x)$ is a solenoid, then $(\overline{Orb(x, f)}, f)$ is mean equicontinuous.

Case 3: $\omega_f(x)$ is infinite and is not a solenoid

Let $f : G \to G$ be a graph map and $x \in G$. Assume that $\omega_f(x)$ is infinite and not a solenoid. Let X be the minimal cycle of graphs containing $\omega_f(x)$ and $E = E(X, f)$. If E is circumferential, then (X, f) is mean equicontinuous. In addition, $(\overline{Orb(x, f)}, f)$ is mean equicontinuous.

Linear disjointness

We say that a sequence $(c_n)_{n=1}^{\infty}$ is linearly disjoint from a dynamical system (X, f) if for any $x \in X$ and any continuous function $\varphi\colon X \to \mathbb{C}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n \varphi(f^n(x)) = 0.$$

Linear disjointness

The Möbius function $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$ is defined as follows: $\mu(1) = 1$, $\mu(n) = (-1)^k$ when n is a product of k distinct primes and $\mu(n) = 0$ otherwise.

Background

Sarnak's Conjecture

The Möbius function $\mu(n)$ is linearly disjoint from all dynamical systems with zero topological entropy.

Ferenczi, S.; Kułaga-Przymus, Joanna; Lemańczyk, M. Sarnak's conjecture: what's new. Ergodic theory and dynamical systems in their interactions with arithmetics and combinatorics, 163–235, Lecture Notes in Math., 2213, Springer, Cham, 2018. MR3821717

Background

Theorem (Fan-Jiang 2018)

Any oscillating sequence is linearly disjoint from all dynamical systems which are minimally mean attractable and minimal mean Lyapunov stable.

Observation

The Möbius function is an oscillating sequence.

Fan, A.; Jiang, Y. Oscillating sequences, MMA and MMLS flows and Sarnak's conjecture. Ergodic Theory Dynam. Systems 38 (2018), no. 5, 1709–1744. MR3819999

Möbius disjointness conjecture

Observation

If a dynamical system (X, f) is locally mean equicontinuous, then it is minimally mean attractable and minimally mean Lyapunov stable.

Corollary (Li-Oprocha-Y.-Zeng 2017)

Any oscillating sequence is linearly disjoint from all continuous graph maps with zero topological entropy. In particular, Sarnak's Möbius disjointness conjecture holds for graph maps with zero topological entropy.

Quasi-graph

Definition

A quasi-graph is a non-degenerate, compact, arcwise connected metric space X satisfying that there is a positive integer N such that $\overline{Y} \setminus Y$ has at most N arcwise connected components for every arcwise connected subset $Y \subset X$.

Theorem (Mai-Shi 2017)

"Roughly speaking", a quasi-graph is a graph with finite Warsaw circles.

Mai. J.; Shi, E. Structures of quasi-graphs and ω-limit sets of quasi-graphs maps. Trans. Amer. Math. Soc. **369** (2017) no. 1, 139–165.

Quasi-graph

Theorem (Li-Oprocha-Zhang, preprint)

Let X be a quasi-graph and let $f\colon X \to X$ be a continuous map. Assume $h(f) > 0$. Then there exist strictly increasing sequences s_n, k_n of positive integers such that each f^{k_n} has an s_n-horseshoe and

$$\lim_{n\to\infty} \frac{1}{k_n} \log(s_n) = h(f).$$

Theorem (Li-Oprocha-Zhang, preprint)

Let X be a quasi-graph and let $f\colon X \to X$ be a continuous map with zero topological entropy. Then every invariant measure of (X, f) has discrete spectrum.

Li, J.; Oprocha, P.; Zhang, G. On dynamics of quasi-graph maps. arXiv:1809.05617

Quasi-graph

Theorem (Huang-Wang-Ye 2019)

Let (X, f) be a dynamical system. If every invariant measure of (X, f) has discrete spectrum, then Sarnak's Möbius disjointness conjecture holds for (X, f).

Corollary (Li-Oprocha-Zhang, preprint)

Sarnak's Möbius disjointness conjecture holds for quasi-graph maps with zero topological entropy.

Huang, W.; Wang, Z.; Ye, X. Measure complexity and Möbius disjointness. Adv. Math. 347 (2019), 827–858. MR3920840

Thank you for your attention.