

# ON THE ROTATION SETS OF HOMEOMORPHISMS OF THE TORUS $\mathbb{T}^d$

Paulo Varandas

UFBA & FCT-CMUP

<https://sites.google.com/view/paulovarandas/>  
(joint with W. Bonomo - UFES and H. Lima - UFBA)

Krakow, Conference on Dynamical Systems Celebrating Michał  
Misiurewicz's 70th Birthday, 2019

# A FLOWER



Daisy (*Bellis perennis*)

*Ergod. Th. & Dynam. Sys.* (First published online 2017), page 1 of 20  
doi:10.1017/etds.2016.109 © Cambridge University Press, 2017

Diffeomorphism without any Measure with  
Maximal Entropy

M. MISUREWICZ

Presented by R. SHIMIZUKI on May 26, 1973

**Summary.** An example which shows that a  $C^0$ -diffeomorphism (y-axis) of a compact manifold (one itself) can have no measure with maximal entropy is given. After a small modification the same example shows that topological entropy is not an upper semi-continuous function of a diffeomorphism in  $C^0$ -topology. A notion of asymptotic  $k$ -expansiveness of a transformation is introduced and it is shown that Buzzi's notion of  $k$ -expansiveness but is even out to be still stronger than it.

PROCEEDINGS OF THE  
AMERICAN MATHEMATICAL SOCIETY  
Volume 120, Number 1, May 1992

## ROTATION SETS OF TORAL FLOWS

BRIEN FRANKS AND MICHAEL MESSERBACH\*

©Copyrighted by Kenneth W. Meyer

**ABSTRACT.** We consider the rotation set  $\rho(\Phi)$  for a lift  $\Phi = [\Phi_t]_{t \in \mathbb{R}}$  of a flow  $\varphi = [\varphi_t: T^2 \rightarrow T^2]_{t \in \mathbb{R}}$ . Our main result is that  $\rho(\Phi)$  consists of either a single point, a segment of a line through  $\mathbb{Q}$  with rational slope, or a line segment with irrational slope and one endpoint equal to  $\mathbb{Q}$ . Any set of one of these types is the rotation set for some flow.

In this article we consider the rotation set as

(1.1) **Definition.** The rotation set  $\rho(\Phi)$  of a flow by  $v \in \rho(\Phi)$  if and only if there are sequences  $\lim_{i \rightarrow \infty} t_i = \infty$  such that

$$\lim_{t \rightarrow \infty} \frac{\Phi_t(x_i) - x_i}{t} = 0$$

## ROTATION SETS FOR MAPS OF TORI

MICHAŁ MISIUREWICZ AND KRYSZYNA ZIEMIAN

## Introduction

The notion of the rotation number of an orientation preserving homeomorphism of a circle was introduced by Poincaré in [8], and since then it has proved to be very useful. It was generalized to the case of continuous maps of a circle of degree one by Newhouse, Palis and Takens in 1979 [7]. In this case one gets a rotation interval. This concept also is very useful. Therefore it seems natural to try to generalize this notion to many-dimensional cases. This idea appears in the papers of Kim, MacKay and Guckenheimer [5], Libbre and MacKay [6] and Herman [3]. Here we shall try to proceed more systematically.

### Counting preimages

MIGUEL MONTENEGRO and ANA RODRIGUES

<sup>2</sup> Department of Mathematical Sciences, RUTGERS, 403 N. Broad Street

For small, non-linearly elastic bodies, the total strain energy can be written as

† Department of Mathematics, University of Exeter, Haverhill Building

E-mail: A.Andriewicz@uwaterloo.ca

(Received 19 January 2016 and accepted in revised form 21 January 2016)

INACT. For non-invertible maps, which are mainly of finite type and piecewise monotone interval maps, we investigate what happens if we follow backward trajectories which are random in the sense that, at each step, every preimage can be chosen with

A SHORT PROOF OF THE VARIATIONAL PRINCIPLE  
FOR A  $\mathbb{Z}^N$  ACTION ON A COMPACT SPACE

by

Michael Misurwicz

0. Introduction. Buelle in [12] introduced the notion of pressure for an action of the group  $\mathbb{Z}^n$  on a compact metric space. It is a generalization of the notion of topological entropy. The variational principle (proved in [12] under some strong conditions) is a generalization of the Minaburg's theorem  $(\Phi, 0, \bar{0})$  on a connection between the topological and measure entropies. A general proof

*Ergod. Th. & Dynam. Sys.* (2006), **26**, 1285–1305 © 2006 Cambridge University Press  
doi:10.1017/S014318570600017X Printed in the United Kingdom

## Affine actions of a free semigroup on the real line

VITALY BERGELSON, MICHAŁ MOSKOWICZ and SAMUEL SENTIN

<sup>2</sup> Department of Mathematics, Ohio State University, Columbus, OH 43210, USA

for email: [vinyl@earthlink.net](mailto:vinyl@earthlink.net) or <http://www.vinyl.com>

<sup>1</sup> Department of Mathematical Sciences, Indiana University Purdue University

Indianapolis, Indianapolis, IN 46202-3216, USA

(e-mail: [monique@math.ignat.edu](mailto:monique@math.ignat.edu))

† Instituto de Matemática Pura e Aplicada, Rio de Janeiro 22460-320, Brazil  
(e-mail: [carstilha@impa.br](mailto:carstilha@impa.br))

(E-mail: [ben@wagyu.be](mailto:ben@wagyu.be))

(Received 13 December 2005 and accepted in revised form 21 May 2006)

**Abstract.** We consider actions of the free semigroup with two generators on the real line where the generators act as affine maps, one contracting and one expanding, with distinct fixed points. Then every orbit is dense in a half-line, which leads to the question whether it is, in some sense, uniformly distributed. We present answers to this question for various interpretations of the phrase ‘uniformly distributed’.

# PLAN OF THE TALK

- Rotation number and rotation sets
- Statement of the main results
- Ideas in the proof(s)

# ROTATION THEORY

$S^1 = \mathbb{R}/\mathbb{Z}$  circle     $\pi : \mathbb{R} \rightarrow S^1$  natural projection

$\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$  circle     $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$  natural projection

Given  $f \in \text{Homeo}_+(\mathbb{S}^1)$ , a lift is a (continuous) map  $F : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\pi \circ F = f \circ \pi$ : thus  $F(x+1) = F(x) + 1$  for all  $x \in \mathbb{R}$

**Poincaré** introduced the translation number

$$\rho(F) = \lim_{n \rightarrow \infty} \frac{F^n(\tilde{x}) - \tilde{x}}{n} \quad (\text{independes of } \tilde{x} \in \mathbb{R})$$

and the rotation number

$$\rho(f) = \rho(F)(\text{mod } 1) \quad (\text{independes of } F)$$

$\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$  circle     $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$  natural projection

Given  $f \in \text{Homeo}_+(\mathbb{S}^1)$ , a lift is a (continuous) map  $F : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\pi \circ F = f \circ \pi$ : thus  $F(x+1) = F(x) + 1$  for all  $x \in \mathbb{R}$

**Poincaré** introduced the **translation number**

$$\rho(F) = \lim_{n \rightarrow \infty} \frac{F^n(\tilde{x}) - \tilde{x}}{n} \quad (\text{independes of } \tilde{x} \in \mathbb{R})$$

and the **rotation number**

$$\rho(f) = \rho(F)(\text{mod } 1) \quad (\text{independes of } F)$$

**Rmk:** The rotation number is effective:

1.  $\rho(f)$  is a topological invariant
2.  $\rho(f) \in \mathbb{Q}$  if and only if  $\text{Per}(f) \neq \emptyset$  (and all have same period)
3. if  $\rho(f) \notin \mathbb{Q}$  then  $\omega(x) \subset \mathbb{S}^1$  is minimal, for all  $x$
4. (Poincaré) if  $\alpha = \rho(f) \notin \mathbb{Q}$  and  $f$  is transitive then  $f$  is topologically conjugated to  $R_\alpha$
5. Rational rotation vector holds open & densely in  $\text{Homeo}_+(\mathbb{S}^1)$



$\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$   $d$ -torus,  $d \geq 2$      $\pi : \mathbb{R}^d \rightarrow \mathbb{T}^d$  natural projection

Given  $f \in \text{Homeo}_0(\mathbb{T}^d)$  isotopic to identity, a lift is a (continuous) map  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $\pi \circ F = f \circ \pi$ :  $F(x + u) = F(x) + u$  for all  $x \in \mathbb{R}^d$  and all  $u \in \mathbb{Z}^d$

$\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$   $d$ -torus,  $d \geq 2$      $\pi : \mathbb{R}^d \rightarrow \mathbb{T}^d$  natural projection

Given  $f \in \text{Homeo}_0(\mathbb{T}^d)$  isotopic to identity, a lift is a (continuous) map  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $\pi \circ F = f \circ \pi$ :  $F(x + u) = F(x) + u$  for all  $x \in \mathbb{R}^d$  and all  $u \in \mathbb{Z}^d$

**Misiurewicz and Ziemian (JLMS 1991)** introduced several **rotation sets** to measure "global" displacement of points in  $\mathbb{T}^d$

**1. Pointwise rotation set** Given  $x \in \mathbb{T}^d$  set

$$\rho(F, x) := \text{accumulation vectors of } \left( \frac{F^n(\tilde{x}) - \tilde{x}}{n} \right)_{n \geq 1}$$

(independ of  $\tilde{x} \in \pi^{-1}(x)$  )

$$\rho_p(F) := \bigcup_{x \in \mathbb{T}^d} \rho(F, x)$$

$\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$   $d$ -torus,  $d \geq 2$      $\pi : \mathbb{R}^d \rightarrow \mathbb{T}^d$  natural projection

Given  $f \in \text{Homeo}_0(\mathbb{T}^d)$  isotopic to identity, a lift is a (continuous) map  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $\pi \circ F = f \circ \pi$ :  $F(x + u) = F(x) + u$  for all  $x \in \mathbb{R}^d$  and all  $u \in \mathbb{Z}^d$

**Misiurewicz and Ziemian (JLMS 1991)** introduced several **rotation sets** to measure "global" displacement of points in  $\mathbb{T}^d$

**1. Pointwise rotation set** Given  $x \in \mathbb{T}^d$  set

$$\rho(F, x) := \text{accumulation vectors of } \left( \frac{F^n(\tilde{x}) - \tilde{x}}{n} \right)_{n \geq 1}$$

(independes of  $\tilde{x} \in \pi^{-1}(x)$  )

$$\rho_p(F) := \bigcup_{x \in \mathbb{T}^d} \rho(F, x)$$

**Rmk:**

- $\rho_p(F, x) \subset \mathbb{R}^d$  is compact and connected (Llibre-Mackay)
- Hard to work:**  $\rho_p(F)$  need not be connected (MZ)
- $\left\{ \int \phi_F d\mu : \mu \in \mathcal{M}_{\text{erg}}(\mathbb{T}^d) \right\} \subset \rho_p(F)$     ( $\phi_F = F - \text{id}$  displacement function)

$\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$   $d$ -torus,  $d \geq 2$

**Misiurewicz and Ziemian (JLMS 1991)** introduced several **rotation sets** to measure "global" displacement of points in  $\mathbb{T}^d$

1. Pointwise rotation set

$$\rho_p(F) := \bigcup_{x \in \mathbb{T}^d} \rho(F, x)$$

2. (Ergodic) measure rotation set

$$\rho_{erg}(F) := \left\{ \int \phi_F d\mu : \mu \in \mathcal{M}_{erg}(\mathbb{T}^d) \right\}$$

$\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$   $d$ -torus,  $d \geq 2$

**Misiurewicz and Ziemian (JLMS 1991)** introduced several **rotation sets** to measure "global" displacement of points in  $\mathbb{T}^d$

1. Pointwise rotation set

$$\rho_p(F) := \bigcup_{x \in \mathbb{T}^d} \rho(F, x)$$

2. (Ergodic) measure rotation set

$$\rho_{erg}(F) := \left\{ \int \phi_F d\mu : \mu \in \mathcal{M}_{erg}(\mathbb{T}^d) \right\}$$

3. Rotation set

$$\rho(F) := \text{accumulation vectors of } \left( \frac{F^{n_i}(\tilde{x}_i) - \tilde{x}_i}{n_i} \right)_{i \geq 1}$$

$\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$   $d$ -torus,  $d \geq 2$

**Misiurewicz and Ziemian (JLMS 1991)** introduced several **rotation sets** to measure "global" displacement of points in  $\mathbb{T}^d$

### 1. Pointwise rotation set

$$\rho_p(F) := \bigcup_{x \in \mathbb{T}^d} \rho(F, x)$$

### 2. (Ergodic) measure rotation set

$$\rho_{erg}(F) := \left\{ \int \phi_F d\mu : \mu \in \mathcal{M}_{erg}(\mathbb{T}^d) \right\}$$

### 3. Rotation set

$$\rho(F) := \text{accumulation vectors of } \left( \frac{F^{n_i}(\tilde{x}_i) - \tilde{x}_i}{n_i} \right)_{i \geq 1}$$

**Rmk:**

1.  $\rho_{erg}(F) \subseteq \rho_p(F) \subseteq \rho(F)$
2. **Good news:**  $\rho(F)$  always connected
3.  $\rho_{inv}(F) = \overline{\rho_{erg}(F)}^{co} = \overline{\rho_p(F)}^{co} = \overline{\rho(F)}^{co}$  convex sets

## SOME QUESTIONS

- What is the **shape of rotation sets**, persistence, and what does it say about the dynamics?
- Can one characterize the **subsets of  $\mathbb{R}^d$  that can be realizable** as rotation sets of homeomorphisms?
- How to characterize the complexity of the set of points with a certain rotation vector (i.e. **level sets**)?
- What about the set of **points with wild historic behavior** (largest non-trivial pointwise rotation set)?

# SOME RESULTS ON REALIZATION

**DIMENSION  $d = 2$**  (Franks 88', 89', Llibre-Mackay 91', Misiurewicz-Ziemian 91', Kwapisz 92')

- $\rho(F)$  is convex
- $\forall K \subset \mathbb{R}^2$  convex there is  $f \in \text{Homeo}_0(\mathbb{T}^2)$  s.t.  $\rho(F) = K$
- $\text{Homeo}_0(\mathbb{T}^2) \ni f \mapsto \rho(F)$  is upper semicontinuous (Hausdorff metric)
- if  $f \in \text{Homeo}_0(\mathbb{T}^2)$  then
  - a.  $(0,0) \in \rho_{\text{erg}}(F) \Rightarrow \text{Fix}(F) \neq \emptyset$
  - b.  $v \in \text{interior}(\rho(F)) \cap \mathbb{Q}^2 \Rightarrow$  exists  $p \in \text{Per}(f)$  s.t.  $\rho(F, p) = v$
  - c.  $v \in \text{extremal}(\rho(F)) \cap \mathbb{Q}^2 \Rightarrow$  exists  $p \in \text{Per}(f)$  s.t.  $\rho(F, p) = v$
- if  $\text{int}(\rho(F)) \neq \emptyset$ , then  $h_{\text{top}}(f) > 0$
- for any connected  $C \subset \rho(F)$  there is  $x \in \mathbb{T}^2$  s.t.  $\rho(F, x) = C$



## SOME RESULTS ON REALIZATION

**DIMENSION  $d = 2$**  (Franks 88', 89', Llibre-Mackay 91', Misiurewicz-Ziemian 91', Kwapisz 92')

- $\rho(F)$  is convex
- $\forall K \subset \mathbb{R}^2$  convex there is  $f \in \text{Homeo}_0(\mathbb{T}^2)$  s.t.  $\rho(F) = K$
- $\text{Homeo}_0(\mathbb{T}^2) \ni f \mapsto \rho(F)$  is upper semicontinuous (Hausdorff metric)
- if  $f \in \text{Homeo}_0(\mathbb{T}^2)$  then
  - a.  $(0,0) \in \rho_{\text{erg}}(F) \Rightarrow \text{Fix}(F) \neq \emptyset$
  - b.  $v \in \text{interior}(\rho(F)) \cap \mathbb{Q}^2 \Rightarrow$  exists  $p \in \text{Per}(f)$  s.t.  $\rho(F, p) = v$
  - c.  $v \in \text{extremal}(\rho(F)) \cap \mathbb{Q}^2 \Rightarrow$  exists  $p \in \text{Per}(f)$  s.t.  $\rho(F, p) = v$
- if  $\text{int}(\rho(F)) \neq \emptyset$ , then  $h_{\text{top}}(f) > 0$
- for any connected  $C \subset \rho(F)$  there is  $x \in \mathbb{T}^2$  s.t.  $\rho(F, x) = C$

**DIMENSION  $d \geq 3$**

- there exists  $f \in \text{Homeo}_0(\mathbb{T}^3)$  so that  $\rho(F)$  is not convex
- ...

## SOME RESULTS ON REALIZATION

**DIMENSION  $d = 2$**  (Franks 88', 89', Llibre-Mackay 91', Misiurewicz-Ziemian 91', Kwapisz 92')

- $\rho(F)$  is convex
- $\forall K \subset \mathbb{R}^2$  convex polygon  $\exists f \in \text{Homeo}_0(\mathbb{T}^2)$  s.t.  $\rho(F) = K$
- $\text{Homeo}_0(\mathbb{T}^2) \ni f \mapsto \rho(F)$  is upper semicontinuous (Hausdorff metric)
- if  $f \in \text{Homeo}_0(\mathbb{T}^2)$  then
  - a.  $(0,0) \in \rho_{\text{erg}}(F) \Rightarrow \text{Fix}(F) \neq \emptyset$
  - b.  $v \in \text{interior}(\rho(F)) \cap \mathbb{Q}^2 \Rightarrow$  exists  $p \in \text{Per}(f)$  s.t.  $\rho(F, p) = v$
  - c.  $v \in \text{extremal}(\rho(F)) \cap \mathbb{Q}^2 \Rightarrow$  exists  $p \in \text{Per}(f)$  s.t.  $\rho(F, p) = v$
- if  $\text{int}(\rho(F)) \neq \emptyset$ , then  $h_{\text{top}}(f) > 0$
- for any connected  $C \subset \rho(F)$  there is  $x \in \mathbb{T}^2$  s.t.  $\rho(F, x) = C$

**DIMENSION  $d \geq 3$**

- there exists  $f \in \text{Homeo}_0(\mathbb{T}^3)$  so that  $\rho(F)$  is *not convex*
- ...

**Rmk:** No characterization of the "size" of points with historic behavior

# MAIN RESULTS

0. SOME DEFINITIONS

I. STRUCTURE OF POINTS WITH HISTORIC  
BEHAVIOR

II. SHAPE & STABILITY IN HIGHER DIMENSION

III. EXTREMAL VECTORS AND MAXIMIZING  
MEASURES

## 0. SOME DEFINITIONS

$(X, d)$  compact metric space,  $f : X \rightarrow X$  continuous

Topological entropy

$$h_{\text{top}}(f) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s(n, \varepsilon)$$

where  $s(n, \varepsilon)$  is maximal cardinality of a  $(n, \varepsilon)$ -separated subset.

## 0. SOME DEFINITIONS

$(X, d)$  compact metric space,  $f : X \rightarrow X$  continuous

Topological entropy

$$h_{\text{top}}(f) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s(n, \varepsilon)$$

where  $s(n, \varepsilon)$  is maximal cardinality of a  $(n, \varepsilon)$ -separated subset.  
If  $Z \subset X$  is  $f$ -invariant, define by a Carathéodory structure

$$h_Z(f, \psi) = \lim_{\varepsilon \rightarrow 0} h_Z(f, \varepsilon)$$

where

$$h_Z(f, \varepsilon) = \inf \{s \in \mathbb{R} : m(Z, s, \varepsilon) = 0\}$$

and

$$M(Z, s, \varepsilon, N) = \inf_{\Gamma} \left\{ \sum_{B \in \Gamma} e^{-s n(B)} \right\},$$

where the infimum is taken over all countable collections  $\Gamma$  by dynamical balls  $B$  of radius  $\varepsilon$  and length  $n(B) \geq N$ .

## 0. SOME DEFINITIONS

**Rmk:**  $C^0$ -generic homeomorphisms have infinite topological entropy (Yano 80')

**Metric mean dimension** (Lindenstrauss and Weiss)

$$\underline{\text{mdim}}_M(f) = \liminf_{\varepsilon \rightarrow 0} \frac{\limsup_{n \rightarrow \infty} \frac{1}{n} \log s(n, \varepsilon)}{-\log \varepsilon} \in [0, \overline{\dim}_B(X, d)]$$

which is non-zero only if  $f$  has infinite topological entropy.

## 0. SOME DEFINITIONS

Given  $f \in \text{Homeo}_0(\mathbb{T}^d)$ , a lift  $F$  and  $x \in \mathbb{T}^d$ , the pointwise rotation set  $\rho(F, x)$  is non-trivial if  $\rho(F, x) \neq \{v\}$  for some  $v \in \mathbb{R}^d$

## 0. SOME DEFINITIONS

Given  $f \in \text{Homeo}_0(\mathbb{T}^d)$ , a lift  $F$  and  $x \in \mathbb{T}^d$ , the pointwise rotation set  $\rho(F, x)$  is non-trivial if  $\rho(F, x) \neq \{v\}$  for some  $v \in \mathbb{R}^d$

We say the pointwise rotation set  $\rho(F, x)$  is wild if

$$\rho(F, x) = \rho_p(F)$$

**Rmk:**

1.  $\rho(F, x) \subseteq \rho_p(F)$  for all  $x \in \mathbb{T}^d$
2. it is necessary that  $\rho_p(F)$  is connected in order to exist points with wild rotation sets



# I. STRUCTURE OF POINTS WITH HISTORIC BEHAVIOR

## THEOREM A

• *Volume preserving*: there exists  $\mathcal{R}_1 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^2)$  Baire residual so that for every  $f \in \mathcal{R}_1$  (and lift  $F$ )

- (i)  $\rho_p(F)$  is connected;
- (ii)  $\{x \in \mathbb{T}^2: \rho(F, x) = \rho_p(F) \text{ is non-trivial}\}$  is Baire residual, has full topological pressure and full metric mean dimension.

# I. STRUCTURE OF POINTS WITH HISTORIC BEHAVIOR

## THEOREM A

• *Volume preserving*: there exists  $\mathcal{R}_1 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^2)$  Baire residual so that for every  $f \in \mathcal{R}_1$  (and lift  $F$ )

- (i)  $\rho_p(F)$  is connected;
- (ii)  $\{x \in \mathbb{T}^2 : \rho(F, x) = \rho_p(F) \text{ is non-trivial}\}$  is Baire residual, has full topological pressure and full metric mean dimension.

• *Non-volume preserving*: there exists a Baire residual subset  $\mathcal{R}_2 \subset \{f \in \text{Homeo}_0(\mathbb{T}^2) : \text{int}(\rho(F)) \neq \emptyset\}$  so that if  $f \in \mathcal{R}_2$ , there exists a positive entropy chain recurrent class  $\Gamma \subset \Omega(f)$  so that

$$\{x \in \Gamma : \rho(F, x) \text{ is non-trivial}\}$$

is a Baire residual, full topological pressure and full metric mean dimension subset of  $\Gamma$ .

## II. SHAPE & STABILITY IN HIGHER DIMENSION

**THEOREM B** For any  $d \geq 2$  there exists a  $C^0$ -open and dense subset of  $\text{Homeo}_0(\mathbb{T}^d)$  (and  $\text{Homeo}_{0,\lambda}(\mathbb{T}^d)$ ) formed by homeomorphisms whose rotation sets are *stable, convex polyhedra* with rational vertices. Moreover, in the volume preserving case the polyhedra have non-empty interior.



**Rmk:** Homeomorphisms with exceptional rotation sets do exist (Misiurewicz-Ziemian 1991')

MICHAŁ MISIUREWICZ AND KRYSZYNA ZIEMIAN

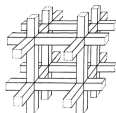


FIG. 44

### III. ERGODIC OPTIMIZATION

**THEOREM C** *Let  $d \geq 2$  there exists a  $C^0$ -open and dense subset of homeomorphisms in  $\text{Homeo}_0(\mathbb{T}^d)$  (and  $\text{Homeo}_{0,\lambda}(\mathbb{T}^d)$ ) so that every extremal vector  $v \in \rho(F)$  is only realizable by periodic points. In particular*

$$H_v := \left\{ x \in \mathbb{T}^d : v \in \rho(F, x) \right\}$$

*has zero topological entropy.*

# IDEAS IN THE PROOF(S)

# IDEAS IN THE PROOF(S)

**THEOREM B (VOLUME PRESERVING)** *For any  $d \geq 2$  there exists a  $C^0$ -open and dense subset of  $\text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  formed by homeomorphisms whose rotation sets are **stable**, **convex polyhedra** with rational vertices and non-empty interior.*

## Step 1: Convexity of rotation sets is $C^0$ -generic

Take  $d \geq 2$  arbitrary (the theorem is known when  $d = 2$ )

**Proposition 1:** There exists a Baire residual  $\mathcal{R}_3 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  so that every  $f \in \mathcal{R}_3$  has convex rotation sets.

## Step 1: Convexity of rotation sets is $C^0$ -generic

Take  $d \geq 2$  arbitrary (the theorem is known when  $d = 2$ )

**Proposition 1:** There exists a Baire residual  $\mathcal{R}_3 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  so that every  $f \in \mathcal{R}_3$  has convex rotation sets.

Proof.

- $C^0$ -generic homeomorphisms satisfy the specification property (Guilheneuf-Lefeuvre)



## Step 1: Convexity of rotation sets is $C^0$ -generic

Take  $d \geq 2$  arbitrary (the theorem is known when  $d = 2$ )

**Proposition 1:** There exists a Baire residual  $\mathcal{R}_3 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  so that every  $f \in \mathcal{R}_3$  has convex rotation sets.

Proof.

- $C^0$ -generic homeomorphisms satisfy the specification property (Guilheneuf-Lefeuvre)
- The set of all periodic measures is dense in  $\mathcal{M}_f(\mathbb{T}^d)$  in the weak\* topology (Sigmund)

## Step 1: Convexity of rotation sets is $C^0$ -generic

Take  $d \geq 2$  arbitrary (the theorem is known when  $d = 2$ )

**Proposition 1:** There exists a Baire residual  $\mathcal{R}_3 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  so that every  $f \in \mathcal{R}_3$  has convex rotation sets.

Proof.

- $C^0$ -generic homeomorphisms satisfy the specification property (Guilheneuf-Lefeuvre)
- The set of all periodic measures is dense in  $\mathcal{M}_f(\mathbb{T}^d)$  in the weak\* topology (Sigmund)
- Since  $\rho_{inv}(F) = \overline{\rho_{erg}(F)}^{co} = \overline{\rho(F)}^{co}$  we conclude that

$$\rho_{inv}(F) = \overline{\rho_{erg}(F)} = \overline{\rho(F)}$$

## Step 1: Convexity of rotation sets is $C^0$ -generic

Take  $d \geq 2$  arbitrary (the theorem is known when  $d = 2$ )

**Proposition 1:** There exists a Baire residual  $\mathcal{R}_3 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  so that every  $f \in \mathcal{R}_3$  has convex rotation sets.

Proof.

- $C^0$ -generic homeomorphisms satisfy the specification property (Guilheneuf-Lefeuvre)
- The set of all periodic measures is dense in  $\mathcal{M}_f(\mathbb{T}^d)$  in the weak\* topology (Sigmund)
- Since  $\rho_{inv}(F) = \overline{\rho_{erg}(F)}^{co} = \overline{\rho(F)}^{co}$  we conclude that

$$\rho_{inv}(F) = \overline{\rho_{erg}(F)} = \overline{\rho(F)} = \rho(F) \quad (\text{the rotation set is compact})$$

- Convexity follows

## Step 2: Stability and realization of extremal rational vectors is $C^0$ -generic

(after Franks 88', Addas-Zanata 04')

**Proposition 2:** There exists a residual subset  $\mathcal{R}_4 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  such that if  $f \in \mathcal{R}$  every extremal vector  $v \in \rho(F)$  is rational and realizable (only) by periodic points.

Proof.

- $C^0$ -generic homeomorphisms satisfy the shadowing property (Kóscielniak-Mazur-Oprocha-Pilarczyk)

## Step 2: Stability and realization of extremal rational vectors is $C^0$ -generic

(after Franks 88', Addas-Zanata 04')

**Proposition 2:** There exists a residual subset  $\mathcal{R}_4 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  such that if  $f \in \mathcal{R}$  every extremal vector  $v \in \rho(F)$  is rational and realizable (only) by periodic points.

Proof.

- $C^0$ -generic homeomorphisms satisfy the shadowing property (Kóscielniak-Mazur-Oprocha-Pilarczyk)
- $f$  satisfies the shadowing property  $\Rightarrow$  the rotation sets are upper-stable: there exists  $\delta > 0$  such that if  $d_{C^0}(F, G) < \delta$  then  $\rho(G) \subseteq \rho(F)$

## Step 2: Stability and realization of extremal rational vectors is $C^0$ -generic

(after Franks 88', Addas-Zanata 04')

**Proposition 2:** There exists a residual subset  $\mathcal{R}_4 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  such that if  $f \in \mathcal{R}$  every extremal vector  $v \in \rho(F)$  is rational and realizable (only) by periodic points.

**Proof.**

- $C^0$ -generic homeomorphisms satisfy the shadowing property (Kóscielniak-Mazur-Oprocha-Pilarczyk)
- $f$  satisfies the shadowing property  $\Rightarrow$  the rotation sets are upper-stable: there exists  $\delta > 0$  such that if  $d_{C^0}(F, G) < \delta$  then  $\rho(G) \subseteq \rho(F)$
- **Key Lemma:** if  $\rho(F)$  is  $\delta$ -upper stable then all extremal vectors are rational and for every  $v \in \rho(F)$  extremal there are no non-atomic probabilities  $\mu$  so that  $\rho(F, \mu) = v$  (uses Atkinson's

## Step 3: From $C^0$ -generic to $C^0$ -open and dense (after Guilheneuf-Koropecski 17')

### Proof of Theorem B (volume preserving).

- It is enough to prove  $C^0$ -denseness of

$$\mathcal{O}_{\mathcal{H}} = \{f \in \text{Homeo}_{0,\lambda}(\mathbb{T}^d) : f \text{ has rotation sets which are stable,} \\ \text{convex polyhedra with non-empty interior}\}$$

## Step 3: From $C^0$ -generic to $C^0$ -open and dense (after Guilheneuf-Koropecski 17')

### Proof of Theorem B (volume preserving).

- It is enough to prove  $C^0$ -denseness of

$$\mathcal{O}_{\mathcal{H}} = \{f \in \text{Homeo}_{0,\lambda}(\mathbb{T}^d) : f \text{ has rotation sets which are stable,} \\ \text{convex polyhedra with non-empty interior}\}$$

- Given  $f \in \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  and  $\varepsilon > 0$  take  $g \in \mathcal{R}_3 \cap \mathcal{R}_4$  that is  $\varepsilon$ - $C^0$ -close to  $f$ ,  $\rho(G)$  is  $\delta$ -upper-stable for some  $\delta > 0$  and every extremal vector  $v \in \rho(F)$  is realizable by periodic points



## Step 3: From $C^0$ -generic to $C^0$ -open and dense (after Guilheneuf-Koropecski 17')

### Proof of Theorem B (volume preserving).

- It is enough to prove  $C^0$ -denseness of

$$\mathcal{O}_{\mathcal{H}} = \{f \in \text{Homeo}_{0,\lambda}(\mathbb{T}^d) : f \text{ has rotation sets which are stable,} \\ \text{convex polyhedra with non-empty interior}\}$$

- Given  $f \in \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  and  $\varepsilon > 0$  take  $g \in \mathcal{R}_3 \cap \mathcal{R}_4$  that is  $\varepsilon$ - $C^0$ -close to  $f$ ,  $\rho(G)$  is  $\delta$ -upper-stable for some  $\delta > 0$  and every extremal vector  $v \in \rho(F)$  is realizable by periodic points
- Extremal vectors are finite (hence  $\rho(G)$  is rational polyhedron) (Guilheneuf-Koropecski 17')

## Step 3: From $C^0$ -generic to $C^0$ -open and dense (after Guilheneuf-Koropecski 17')

### Proof of Theorem B (volume preserving).

- It is enough to prove  $C^0$ -denseness of

$\mathcal{O}_{\mathcal{H}} = \{f \in \text{Homeo}_{0,\lambda}(\mathbb{T}^d) : f \text{ has rotation sets which are stable, convex polyhedra with non-empty interior}\}$

- Given  $f \in \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$  and  $\varepsilon > 0$  take  $g \in \mathcal{R}_3 \cap \mathcal{R}_4$  that is  $\varepsilon$ - $C^0$ -close to  $f$ ,  $\rho(G)$  is  $\delta$ -upper-stable for some  $\delta > 0$  and every extremal vector  $v \in \rho(F)$  is realizable by periodic points
- Extremal vectors are finite (hence  $\rho(G)$  is rational polyhedron) (Guilheneuf-Koropecski 17')
- Take  $\varepsilon$ - $C^0$ -small perturbation  $H$  of  $G$  such that "boundary" periodic points are preserved and become stable

## Step 3: From $C^0$ -generic to $C^0$ -open and dense

Proof of Theorem B (volume preserving).

- It is enough to prove  $C^0$ -denseness of

$$\mathcal{O}_{\mathcal{H}} = \{f \in \text{Homeo}_{0,\lambda}(\mathbb{T}^d) : f \text{ has rotation sets which are stable,} \\ \text{convex polyhedra with non-empty interior}\}$$

- $\rho(G)$  has non-empty interior: otherwise, take a suitable perturbation  $T_v \circ G$  of  $G$  in such a way that

$$\rho(T_v \circ G, \lambda) = v + \rho(G, \lambda) \notin \rho(G)$$

(contradiction with stability)



# HAPPY BIRTHDAY MICHAŁ



**“Mathematics is the most beautiful and most powerful creation of the human spirit- *Stephan Banach***