On the rotation sets of homeomorphisms of the torus $\mathbb{T}^d$

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(joint with W. Bonomo - UFES and H. Lima - UFBA)

Krakow, Conference on Dynamical Systems Celebrating Michał Misiurewicz's 70th Birthday, 2019

A Flower

Daisy (*Bellis perennis*)

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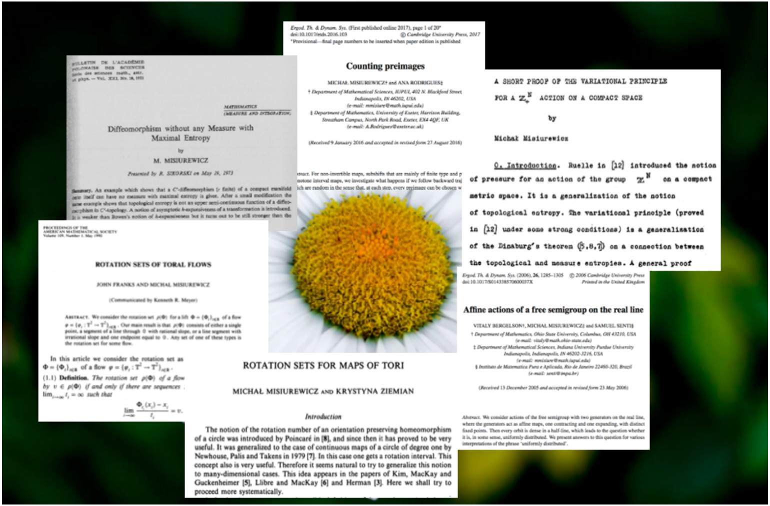

MM Daisy (*Bellis Michał Mathematica Perennis*)

Plan of the talk

- Rotation number and rotation sets
- Statement of the main results
- Ideas in the proof(s)

Rotation Theory

$\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ circle $\quad \pi : \mathbb{R} \to \mathbb{S}^1$ natural projection

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Given $f \in Homeo_+(\mathbb{S}^1)$, a lift is a (continuous) map $F : \mathbb{R} \to \mathbb{R}$ such that $\pi \circ F = f \circ \pi$: thus $F(x+1) = F(x) + 1$ for all $x \in \mathbb{R}$

Poincaré introduced the translation number

$$\rho(F) = \lim_{n\to\infty} \frac{F^n(\tilde{x}) - \tilde{x}}{n} \quad \text{(independs of } \tilde{x} \in \mathbb{R})$$

and the rotation number

$$\rho(f) = \rho(F)(mod 1) \quad \text{(independs of } F)$$

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Rmk: The rotation number is effective:

1. $\rho(f)$ is a topological invariant
2. $\rho(f) \in \mathbb{Q}$ if and only if $Per(f) \neq \emptyset$ (and all have same period)
3. if $\rho(f) \notin \mathbb{Q}$ then $\omega(x) \subset \mathbb{S}^1$ is minimal, for all x
4. (Poincaré) if $\alpha = \rho(f) \notin \mathbb{Q}$ and f is transitive then f is topologically conjugated to R_α
5. Rational rotation vector holds open & densely in $Homeo_+(\mathbb{S}^1)$

$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ d-torus, $d \geq 2$ $\quad \pi : \mathbb{R}^d \to \mathbb{T}^d$ natural projection

Given $f \in Homeo_0(\mathbb{T}^d)$ isotopic to identity, a lift is a (continuous) map $F : \mathbb{R}^d \to \mathbb{R}^d$ such that $\pi \circ F = f \circ \pi$: $F(x+u) = F(x) + u$ for all $x \in \mathbb{R}^d$ and all $u \in \mathbb{Z}^d$

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Misiurewicz and Ziemian (JLMS 1991) introduced several rotation sets to measure "global" displacement of points in $\mathbb{T}^d$

1. Pointwise rotation set Given $x \in \mathbb{T}^d$ set

$$\rho(F, x) := \text{accumulation vectors of } \left(\frac{F^n(\tilde{x}) - \tilde{x}}{n}\right)_{n \geq 1}$$

(independs of $\tilde{x} \in \pi^{-1}(x)$)

$$\rho_p(F) := \bigcup_{x \in \mathbb{T}^d} \rho(F, x)$$

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Rmk:

1. $\rho_p(F,x) \subset \mathbb{R}^d$ is compact and connected (Llibre-Mackay)
2. Hard to work: $\rho_p(F)$ need not be connected (MZ)
3. $\{\int \phi_F \, d\mu : \mu \in \mathcal{M}_{erg}(\mathbb{T}^d)\} \subset \rho_p(F)$ $\quad$ ($\phi_F = F - id$ displacement function)

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2. (Ergodic) measure rotation set

$$\rho_{erg}(F) := \left\{ \int \phi_F \, d\mu : \mu \in \mathcal{M}_{erg}(\mathbb{T}^d) \right\}$$

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Rmk:

1. $\rho_{erg}(F) \subseteq \rho_p(F) \subseteq \rho(F)$
2. Good news: $\rho(F)$ always connected
3. $\rho_{inv}(F) = \overline{\rho_{erg}(F)}^{co} = \overline{\rho_p(F)}^{co} = \overline{\rho(F)}^{co}$ convex sets

Some questions

- What is the shape of rotation sets, persistence, and what does it say about the dynamics?
- Can one characterize the subsets of $\mathbb{R}^d$ that can be realizable as rotation sets of homeomorphisms?
- How to characterize the complexity of the set of points with a certain rotation vector (i.e. level sets)?
- What about the set of points with wild historic behavior (largest non-trivial pointwise rotation set)?

Some results on realization

Dimension $d = 2$ (Franks 88', 89', Llibre-Mackay 91', Misiurewicz-Ziemian 91', Kwapisz 92')

- $\rho(F)$ is convex
- $\forall K \subset \mathbb{R}^2$ convex there is $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ s.t. $\rho(F) = K$
- $\mathrm{Homeo}_0(\mathbb{T}^2) \ni f \mapsto \rho(F)$ is upper semicontinuous (Hausdorff metric)
- if $f \in \mathrm{Homeo}_0(\mathbb{T}^2)$ then
  a. $(0,0) \in \rho_{erg}(F) \Rightarrow Fix(F) \neq \emptyset$
  b. $v \in interior(\rho(F)) \cap \mathbb{Q}^2 \Rightarrow$ exists $p \in Per(f)$ s.t. $\rho(F,p) = v$
  c. $v \in extremal(\rho(F)) \cap \mathbb{Q}^2 \Rightarrow$ exists $p \in Per(f)$ s.t. $\rho(F,p) = v$
- if $\mathrm{int}(\rho(F)) \neq \emptyset$, then $h_{top}(f) > 0$
- for any connected $C \subset \rho(F)$ there is $x \in \mathbb{T}^2$ s.t. $\rho(F,x) = C$

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Dimension $d \geq 3$

- there exists $f \in \mathrm{Homeo}_0(\mathbb{T}^3)$ so that $\rho(F)$ is not convex
- ...

Some results on realization

Dimension $d=2$ (Franks 88', 89', Llibre-Mackay 91', Misiurewicz-Ziemian 91', Kwapisz 92')

- $\rho(F)$ is convex
- $\forall K \subset \mathbb{R}^2$ convex polygon $\exists f \in \text{Homeo}_0(\mathbb{T}^2)$ s.t. $\rho(F) = K$
- $\text{Homeo}_0(\mathbb{T}^2) \ni f \mapsto \rho(F)$ is upper semicontinuous (Hausdorff metric)
- if $f \in \text{Homeo}_0(\mathbb{T}^2)$ then
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Dimension $d \geq 3$

- there exists $f \in \text{Homeo}_0(\mathbb{T}^3)$ so that $\rho(F)$ is *not convex*
- ...

Rmk: No characterization of the "size" of points with historic behavior

Main results

0. Some definitions

I. Structure of points with historic behavior

II. Shape & stability in higher dimension

III. Extremal vectors and maximizing measures

0. Some definitions

(X, d) compact metric space, $\quad f : X \to X$ continuous

Topological entropy

$$h_{\text{top}}(f) \;= \lim_{\varepsilon \to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(n, \varepsilon)$$

where $s(n, \varepsilon)$ is maximal cardinality of a (n, ε)-separated subset.

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If $Z \subset X$ is f-invariant, define by a Carathéodory structure

$$h_Z(f, \psi) = \lim_{\varepsilon\to 0} h_Z(f, \varepsilon)$$

where

$$h_Z(f, \varepsilon) = \inf\{s \in \mathbb{R} : m(Z, s, \varepsilon) = 0\}$$

and

$$M(Z, s, \varepsilon, N) = \inf_{\Gamma} \left\{ \sum_{B\in\Gamma} e^{-s\, n(B)} \right\},$$

where the infimum is taken over all countable collections Γ by dynamical balls B of radius ε and length $n(B) \geq N$.

Rmk: C^0-generic homeomorphisms have infinite topological entropy (Yano 80')

Metric mean dimension (Lindenstrauss and Weiss)

$$\underline{\mathrm{mdim}}_M(f) = \liminf_{\varepsilon \to 0} \frac{\limsup_{n\to\infty} \frac{1}{n} \log s(n,\varepsilon)}{-\log \varepsilon} \in [0, \overline{dim_B}(X,d)]$$

which is non-zero only if f has infinite topological entropy.

0. Some definitions

Given $f \in \mathrm{Homeo}_0(\mathbb{T}^d)$, a lift F and $x \in \mathbb{T}^d$, the pointwise rotation set $\rho(F,x)$ is non-trivial if $\rho(F,x) \neq \{v\}$ for some $v \in \mathbb{R}^d$

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Given $f \in \text{Homeo}_0(\mathbb{T}^d)$, a lift F and $x \in \mathbb{T}^d$, the pointwise rotation set $\rho(F,x)$ is non-trivial if $\rho(F,x) \neq \{v\}$ for some $v \in \mathbb{R}^d$

We say the pointwise rotation set $\rho(F,x)$ is wild if

$$\rho(F,x) = \rho_p(F)$$

Rmk:

1. $\rho(F,x) \subseteq \rho_p(F)$ for all $x \in \mathbb{T}^d$
2. it is necessary that $\rho_p(F)$ is connected in order to exist points with wild rotation sets

I. Structure of points with historic behavior

Theorem A

• *Volume preserving:* there exists $\mathcal{R}_1 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^2)$ Baire residual so that for every $f \in \mathcal{R}_1$ (and lift F)

(i) $\rho_p(F)$ is connected;

(ii) $\{x \in \mathbb{T}^2 \colon \rho(F,x) = \rho_p(F)$ is non-trivial$\}$ is Baire residual, has full topological pressure and full metric mean dimension.

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• *Non-volume preserving:* there exists a Baire residual subset $\mathcal{R}_2 \subset \left\{f \in \text{Homeo}_0(\mathbb{T}^2) : \text{int}(\rho(F)) \neq \emptyset\right\}$ so that if $f \in \mathcal{R}_2$, there exists a positive entropy chain recurrent class $\Gamma \subset \Omega(f)$ so that

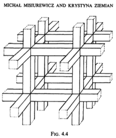
$$\{x \in \Gamma \colon \rho(F,x) \text{ is non-trivial}\}$$

is a Baire residual, full topological pressure and full metric mean dimension subset of Γ.

II. Shape & stability in higher dimension

Theorem B *For any $d \geq 2$ there exists a C^0-open and dense subset of $Homeo_0(\mathbb{T}^d)$ (and $Homeo_{0,\lambda}(\mathbb{T}^d)$) formed by homeomorphisms whose rotation sets are stable, convex polyhedra with rational vertices. Moreover, in the volume preserving case the polyhedra have non-empty interior.*

Rmk: Homeomorphisms with exceptional rotation sets do exist (Misiurewicz-Ziemian 1991')



Theorem C *Let $d \geq 2$ there exists a C^0-open and dense subset of homeomorphisms in $Homeo_0(\mathbb{T}^d)$ (and $Homeo_{0,\lambda}(\mathbb{T}^d)$) so that every extremal vector $v \in \rho(F)$ is only realizable by periodic points. In particular*

$$H_v := \left\{ x \in \mathbb{T}^d : v \in \rho(F, x) \right\}$$

has zero topological entropy.

Ideas in the proof(s)

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THEOREM B (VOLUME PRESERVING) *For any $d \geq 2$ there exists a C^0-open and dense subset of $Homeo_{0,\lambda}(\mathbb{T}^d)$ formed by homeomorphisms whose rotation sets are stable, convex polyhedra with rational vertices and non-empty interior.*

Step 1: Convexity of rotation sets is C^0-generic

Take $d \geq 2$ arbitrary (the theorem is known when $d = 2$)

Proposition 1: There exists a Baire residual $\mathcal{R}_3 \subset \mathrm{Homeo}_{0,\lambda}(\mathbb{T}^d)$ so that every $f \in \mathcal{R}_3$ has convex rotation sets.

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$$\rho_{inv}(F) = \overline{\rho_{erg}(F)} = \overline{\rho(F)} = \rho(F) \quad \text{(the rotation set is compact)}$$

- Convexity follows

Step 2: Stability and realization of extremal rational vectors is C^0-generic

(after Franks 88', Addas-Zanata 04')

Proposition 2: There exists a residual subset $\mathcal{R}_4 \subset \text{Homeo}_{0,\lambda}(\mathbb{T}^d)$ such that if $f \in \mathcal{R}$ every extremal vector $v \in \rho(F)$ is rational and realizable (only) by periodic points.

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- C^0-generic homeomorphisms satisfy the shadowing property (Kóscielniak-Mazur-Oprocha-Pilarczyk)
- f satisfies the shadowing property $\Rightarrow$ the rotation sets are upper-stable: there exists $\delta > 0$ such that if $d_{C^0}(F, G) < \delta$ then $\rho(G) \subseteq \rho(F)$

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- f satisfies the shadowing property $\Rightarrow$ the rotation sets are upper-stable: there exists $\delta > 0$ such that if $d_{C^0}(F, G) < \delta$ then $\rho(G) \subseteq \rho(F)$
- **Key Lemma:** if $\rho(F)$ is δ-upper stable then all extremal vectors are rational and for every $v \in \rho(F)$ extremal there are no non-atomic probabilities μ so that $\rho(F, \mu) = v$ (uses Atkinson's lemma & Hahn-Banach geometric theorem)

Step 3: From C^0-generic to C^0-open and dense

(after Guilheneuf-Koropecki 17')

Proof of Theorem B (volume preserving).

- It is enough to prove C^0-denseness of

$$\mathcal{O}_{\mathcal{H}} = \{f \in \mathrm{Homeo}_{0,\lambda}(\mathbb{T}^d) \colon f \text{ has rotation sets which are stable, convex polyhedra with non-empty interior}\}$$

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- Given $f \in \mathrm{Homeo}_{0,\lambda}(\mathbb{T}^d)$ and $\varepsilon > 0$ take $g \in \mathcal{R}_3 \cap \mathcal{R}_4$ that is ε-C^0-close to f, $\rho(G)$ is δ-upper-stable for some $\delta > 0$ and every extremal vector $v \in \rho(F)$ is realizable by periodic points

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- Extremal vectors are finite (hence $\rho(G)$ is rational polyhedron) (Guilheneuf-Koropecki 17')

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- Extremal vectors are finite (hence $\rho(G)$ is rational polyhedron) (Guilheneuf-Koropecki 17')
- Take ε-C^0-small perturbation H of G such that "boundary" periodic points are preserved and become stable

Step 3: From C^0-generic to C^0-open and dense

Proof of Theorem B (volume preserving).

- It is enough to prove C^0-denseness of

$$\mathcal{O}_{\mathcal{H}} = \{f \in \mathrm{Homeo}_{0,\lambda}(\mathbb{T}^d) \colon f \text{ has rotation sets which are stable,}$$
$$\text{convex polyhedra with non-empty interior}\}$$

- $\rho(G)$ has non-empty interior: otherwise, take a suitable perturbation $T_v \circ G$ of G in such a way that

$$\rho(T_v \circ G, \lambda) = v + \rho(G, \lambda) \notin \rho(G)$$

(contradition with stability)

□

Happy Birthday Michał

“**Mathematics is the most beautiful and most powerful creation of the human spirit-** *Stephan Banach*