Singular stationary measures for Alsedà-Misiurewicz systems

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University of Warsaw

joint work with Krzysztof Barański

Conference on Dynamical Systems
Celebrating Michał Misiurewicz's 70th Birthday

Kraków, June 10, 2019

Ambrosia Mexicana

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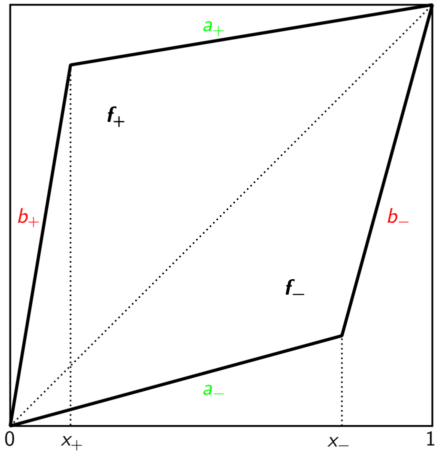
More or less edible



Definition (Alsedà-Misiurewicz systems)

An **AM-system** is the system $\{f_-, f_+\}$ of increasing homeomorphisms of the interval $[0,1]$ of the form

$$f_-(x) = \begin{cases} a_- x & \text{for } x \in [0, x_-] \\ 1 - b_-(1-x) & \text{for } x \in (x_-, 1] \end{cases}$$

$$f_+(x) = \begin{cases} b_+ x, & x \in [0, x_+] \\ 1 - a_+(1-x), & x \in (x_+, 1] \end{cases}$$

where

$$0 < a_- < 1 < b_-,$$

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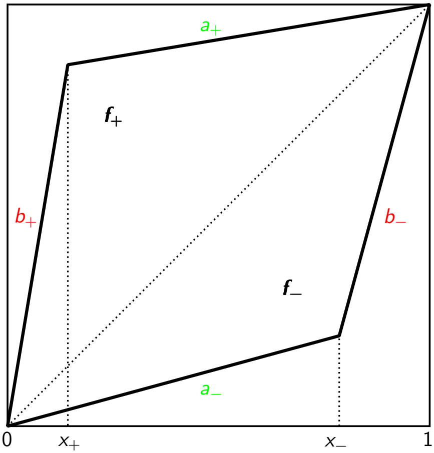
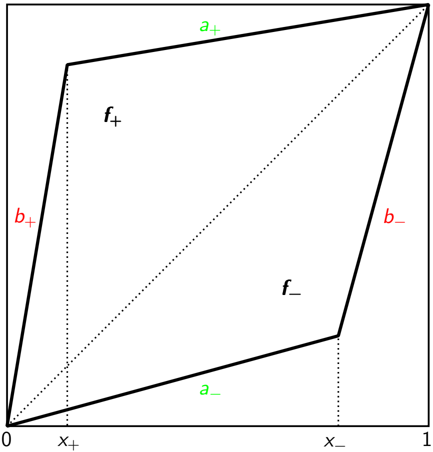
and

$$x_- = \frac{b_- - 1}{b_- - a_-},$$

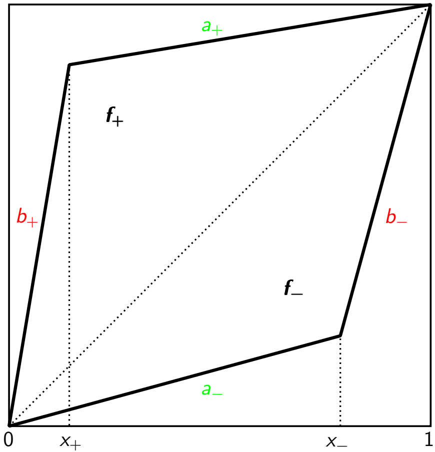
$$x_+ = \frac{1 - a_+}{b_+ - a_+}.$$



a_+
f_+
b_+
b_-
f_-
a_-
0
x_+
x_-
1



(p_-, p_+) - probability vector.



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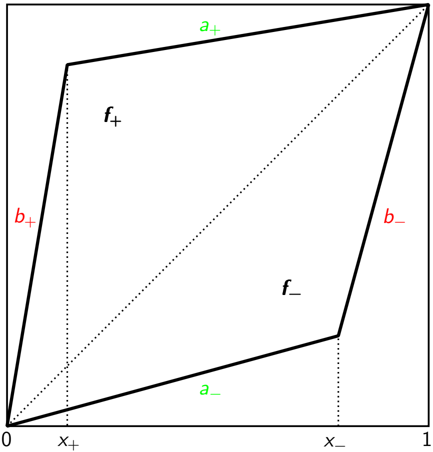

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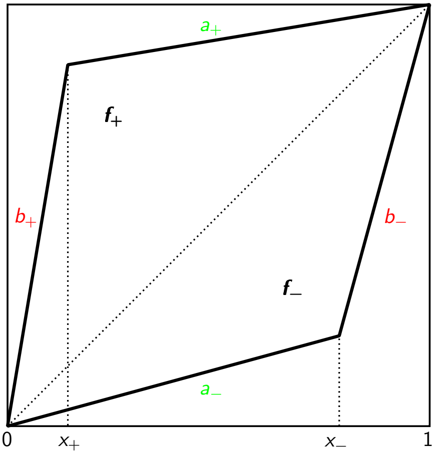
$$\mu = p_-(f_-)_*\mu + p_+(f_+)_*\mu$$

and $\mu(\{0,1\}) = 0.$



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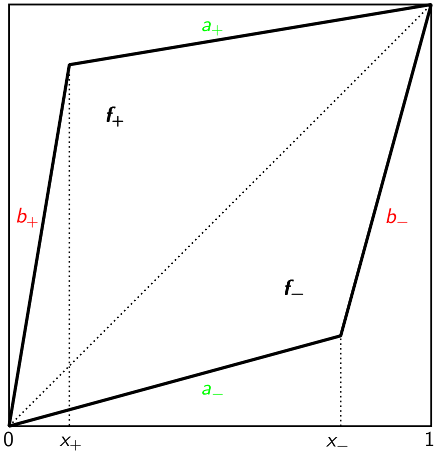

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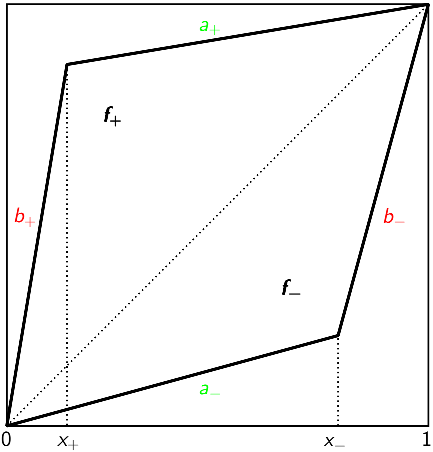

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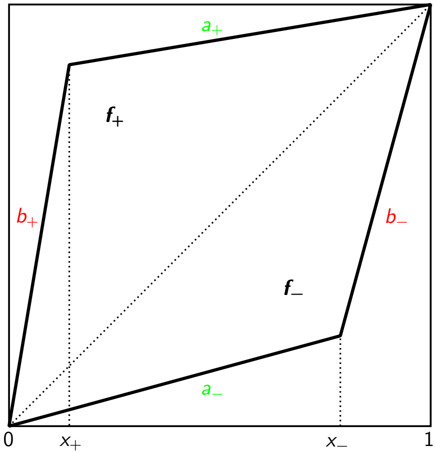

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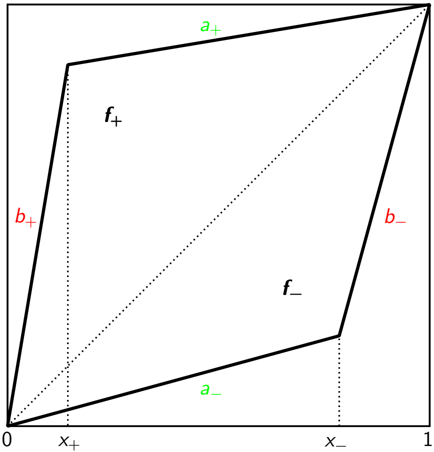

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Results: Singularity and dimension calculation/bounds for certain sets of parameters.

Groups of circle diffeomorphisms

Navas, Question 18[1]: Singularity vs absolute continuity for finitely-generated groups G of C^2 orientation-preserving circle diffeomorphisms.

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Navas, Question 16: Assume that G admits an exceptional minimal set Λ. Is the restriction of the action G to Λ topologically conjugated to the action of a group of piecewise-affine homeomorphisms? (conjecture of Dippolito)

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Definition

We say that an AM-system $\{f_-, f_+\}$ is of:

- **disjoint type**, if the intervals $[0, f_-(x_-)]$, $[f_+(x_+), 1]$ are disjoint, i.e. $f_-(x_-) < f_+(x_+)$,
- **border type**, if the intervals $[0, f_-(x_-)]$, $[f_+(x_+), 1]$ touch each other, i.e. $f_-(x_-) = f_+(x_+)$,
- **overlapping type**, if the intervals $[0, f_-(x_-)]$, $[f_+(x_+), 1]$ overlap, i.e. $f_-(x_-) > f_+(x_+)$.

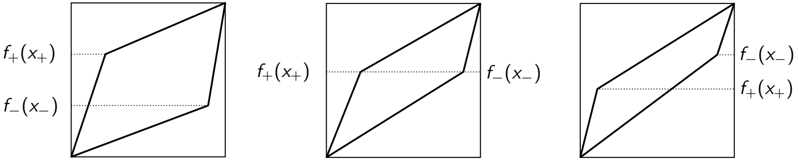

Figure: Three types of AM-systems: disjoint, border and overlapping.

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Theorem 1

Let $\{f_-, f_+\}$ be an AM-system with probabilities p_-, p_+, such that the Lyapunov exponents $\Lambda(0), \Lambda(1)$ are positive. Then the stationary measure μ is the Lebesgue measure on $[0,1]$ if and only if the system is of border type and

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For symmetric systems with $p_- = p_+ = \frac{1}{2}$ this gives $\mu = \mathrm{Leb}\,|_{[0,1]}$ if and only if $f_+(x_+) = f_-(x_-) = \frac{1}{2}$.

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Analogously, a $(k:l)$-resonance at 1 occurs if $\frac{\ln f_-'(1)}{\ln f_+'(1)} = -\frac{k}{l}$. Without loss of generality, we always assume that k, l are relatively prime.

Proposition

If an AM-system with positive Lyapunov exponents has no resonance at one of the endpoints $0, 1$, then it is minimal in $(0, 1)$ and the support of μ is equal to $[0, 1]$.

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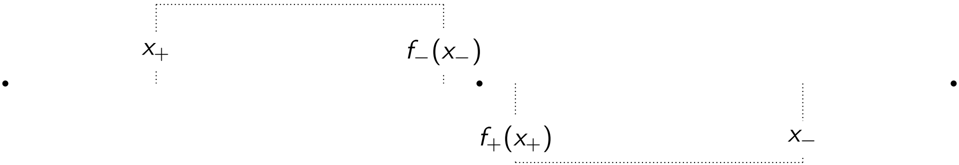

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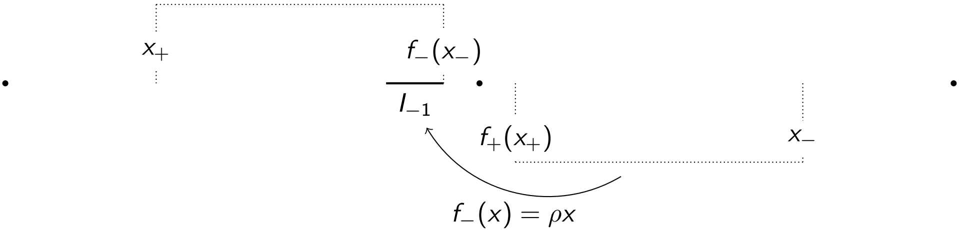

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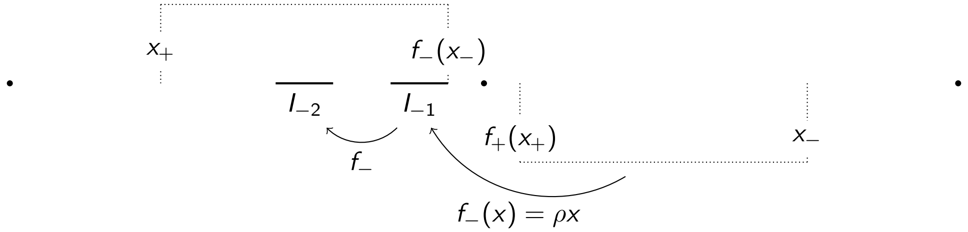

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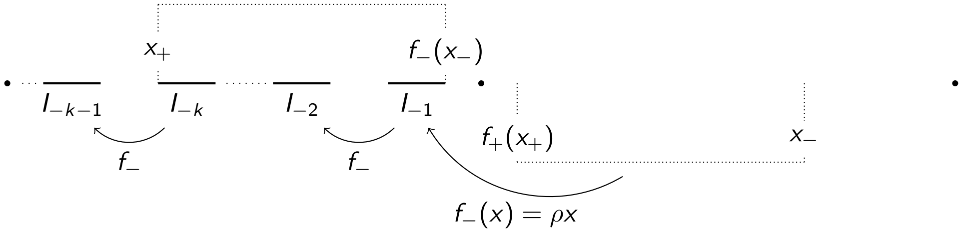

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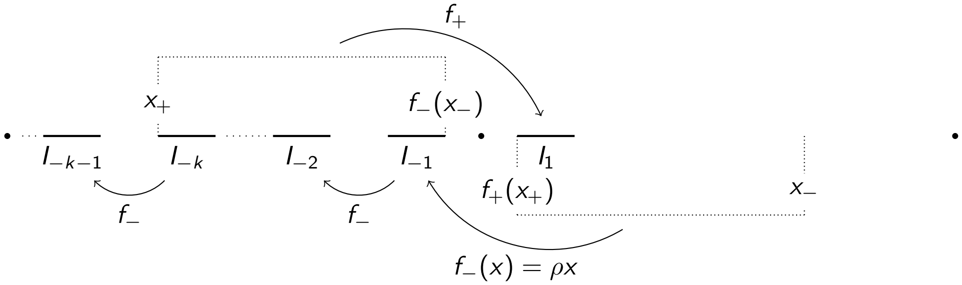

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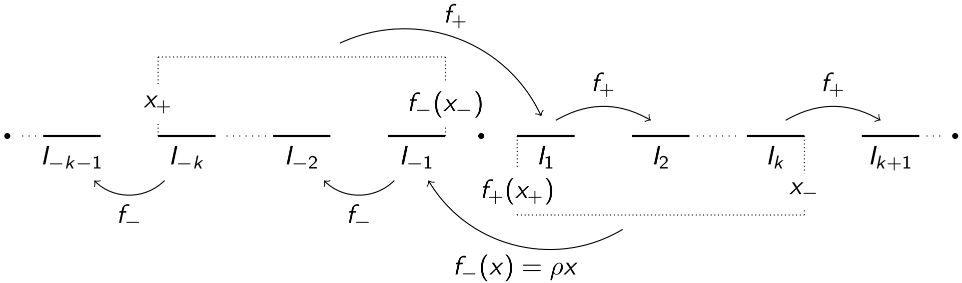

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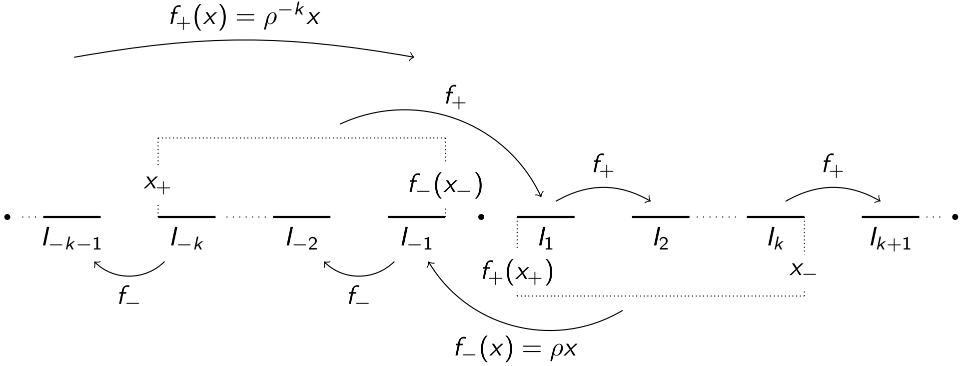

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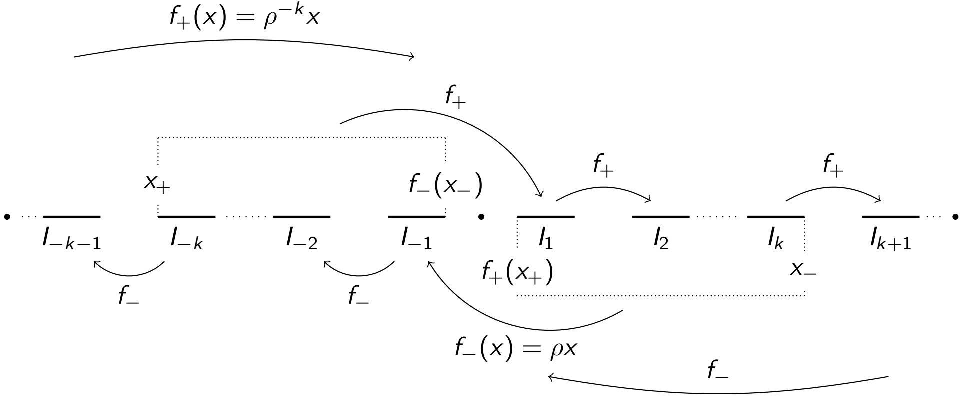

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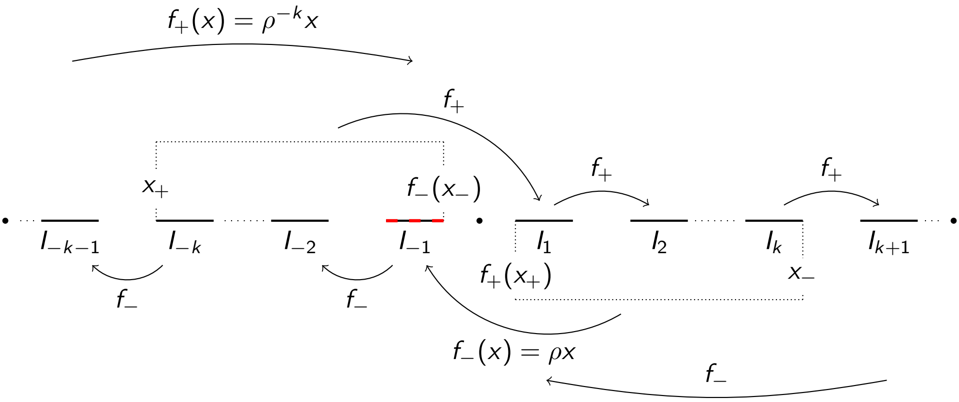

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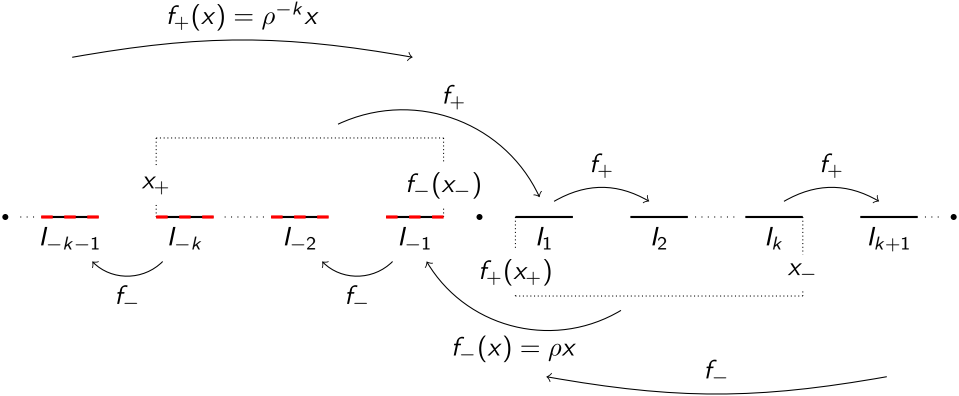

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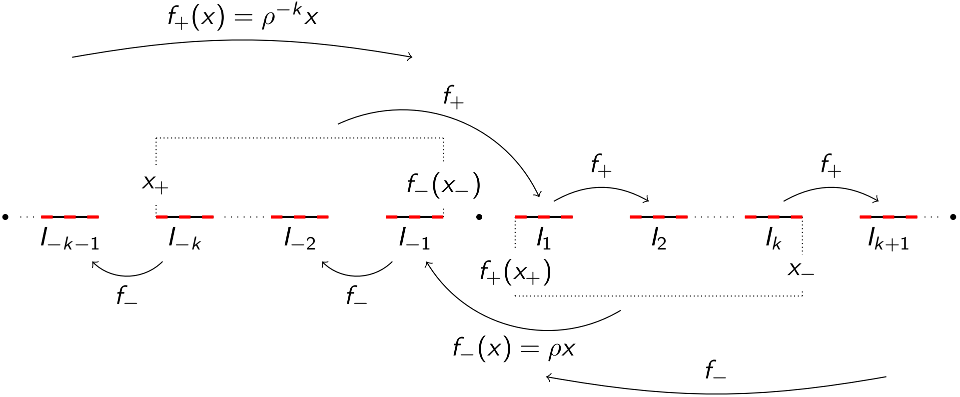

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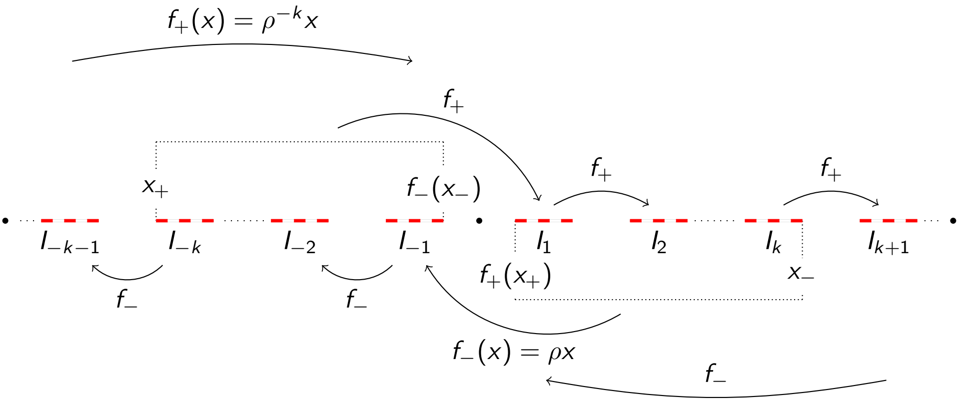

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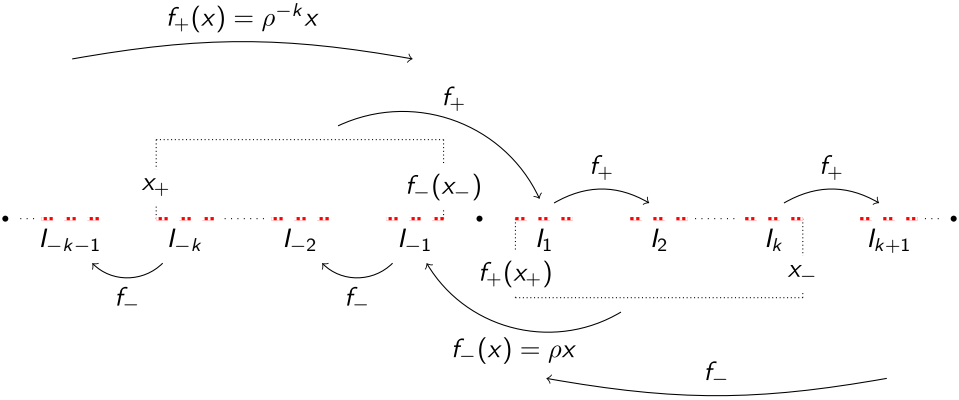

Λ = union of Cantor sets.

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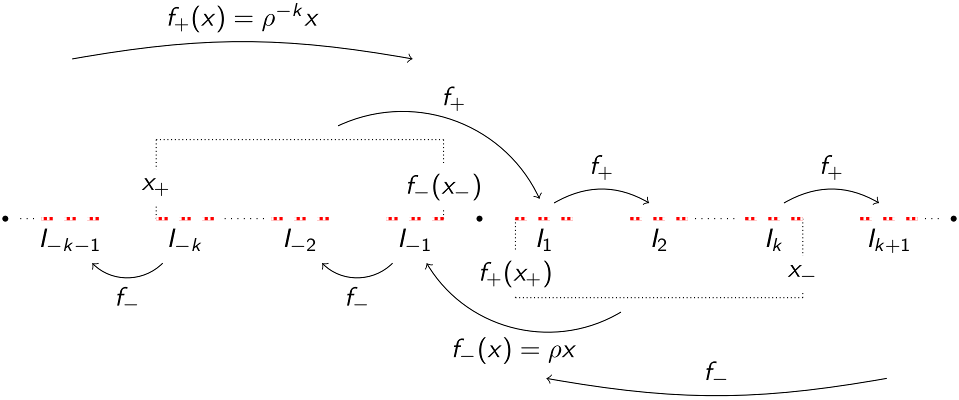

$\Lambda =$ union of Cantor sets. **Claim:** $\overline{\Lambda} = \Lambda \cup \{0,1\} = \omega_\infty(x)$ for $x \in (0,1)$

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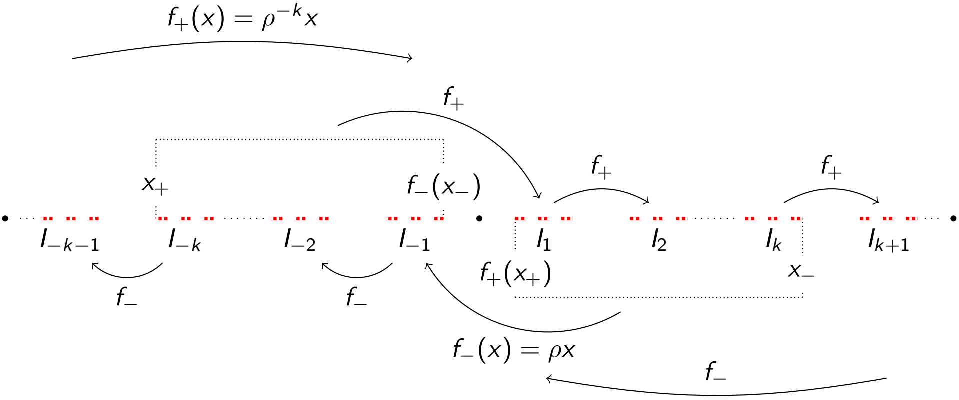

Λ = union of Cantor sets. **Claim:** $\overline{\Lambda} = \Lambda \cup \{0, 1\} = \omega_\infty(x)$ for $x \in (0, 1)$

$\omega_\infty(x) = \{$limit points of trajectiories of x jumping over the interval $(f_-(x_-), f_+(x_+))$ infinitely many times$\}$

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$$\dim_H \mu = \frac{\sum_{r=1}^{k} r\left(\frac{p_+}{p_-}\eta_-^r \log \eta_- + \frac{p_-}{p_+}\eta_+^r \log \eta_+\right)}{\sum_{r=1}^{k} r\left(\frac{p_+}{p_-}\eta_-^r + \frac{p_-}{p_+}\eta_+^r\right)\log \rho},$$

where $\eta_-, \eta_+ \in (0,1)$ are, respectively, the unique solutions of the equations

$$p_+\eta_-^{k+1} - \eta_- + p_- = 0, \qquad p_-\eta_+^{k+1} - \eta_+ + p_+ = 0.$$

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In particular, if $p_- = p_+ = 1/2$, then

$$\dim_H \mu = \dim_H(\operatorname{supp} \mu) = \frac{\log \eta}{\log \rho} < 1.$$

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For $p_- = p_+ = \frac{1}{2}$, measure μ on each of these Cantor sets is (up to normalization) a self-similar measure.

For symmetric systems with $p_- = p_+ = 1/2$ and $l = 1$, we have $\rho = \eta$ if and only if $\mu = \mathrm{Leb}\big|_{[0,1]}$.

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If a symmetric AM-system with probabilities $p_- = p_+ = 1/2$ and positive Lyapunov exponents exhibits $(5:2)$-resonance and satisfies $\rho = \eta$, then μ is singular with

$$\dim_H \mu < 1, \qquad \operatorname{supp} \mu = [0,1].$$

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$$g_- \circ h = h \circ f_-, \qquad g_+ \circ h = h \circ f_+.$$

Consider now the following family of symmetric AM-systems:

$$f_-(x) = \begin{cases} \rho x & , x \in [0, x_-] \\ \rho^{-\gamma} x + 1 - \rho^{-\gamma} & , x \in (x_-, 1] \end{cases},$$

$$f_+(x) = \begin{cases} \rho^{-\gamma} x & \text{for } x \in [0, x_+] \\ \rho x + 1 - \rho & \text{for } x \in (x_+, 1] \end{cases},$$

where $\rho \in (0,1)$, $\gamma > 1$ and

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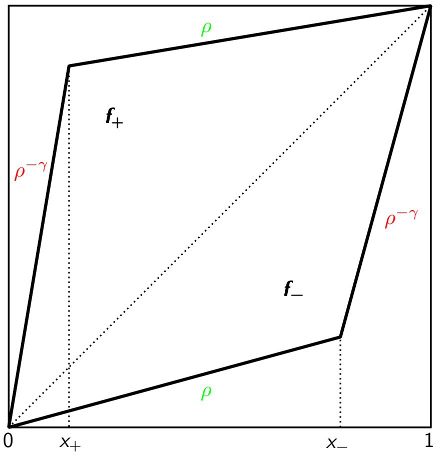

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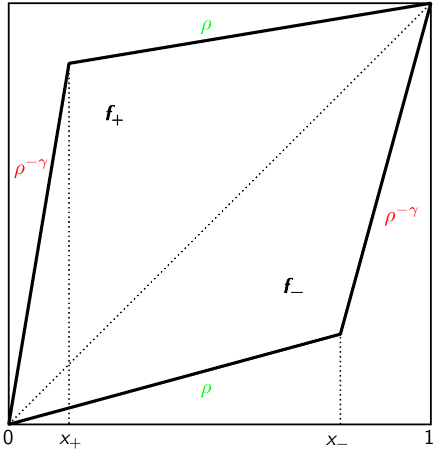

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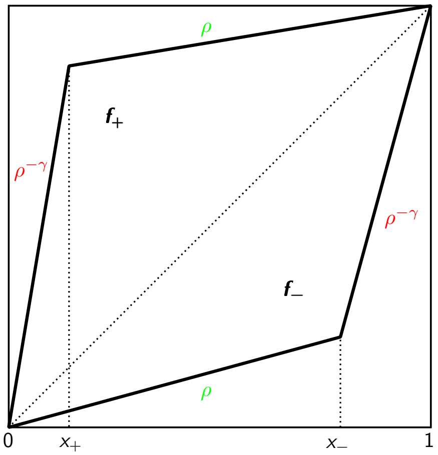

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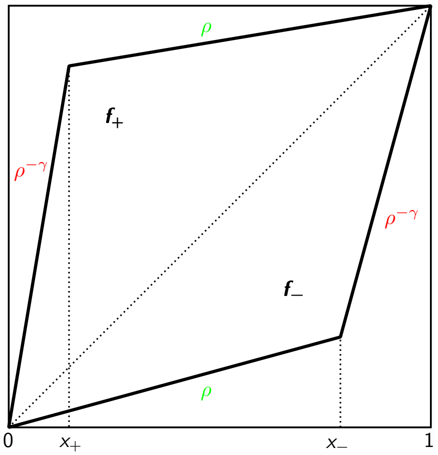

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Let $\mu_{\rho,\gamma}$ be the corresponding stationary measure.

Theorem 6 (work in progress)

Fix $\gamma \in (1, \frac{3}{2})$. For ρ sufficiently small, the corresponding measure $\mu_{\rho,\gamma}$ is singular with $\dim_H(\mu_{\rho,\gamma}) < 1$.

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For a probability vector (p_-, p_+), let

$$H((p_-, p_+)) := -p_- \log p_- - p_+ \log p_+,$$

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We use the following general upper bound:

$$\dim_H(\mu_{\rho,\gamma}) \leq -\frac{H((p_-, p_+))}{\chi(\mu_{\rho,\gamma})},$$

provided $\chi(\mu_{\rho,\gamma}) < 0$.

In our case

$$\chi(\mu_{\rho,\gamma}) = (\frac{1-\gamma}{2} + \mu_{\rho,\gamma}(M)\frac{1+\gamma}{2}) \log \rho,$$

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This can be done, as outside of M the system behaves as a random walk.

Thank you for your attention!