Conference on Dynamical Systems
Celebrating Michal Misiurewicz's 70th Birthday

Fluctuations of Ergodic Sums on Periodic Orbits under Specification

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Happy 70 Michal

Let $(X, T, \mathcal{B}, \mu)$ be a probability space:

- $T : X \to X$ a measurable, expansive transformation of a compact metric space X.
- $\mathcal{B}$ the Borel σ-algebra.
- μ a T-invariant ergodic probability.

For any observable function $\varphi : X \to \mathbb{R}$ denote the Cesàro sum by

$$\frac{1}{n} S_0^n \varphi(x) := \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i x)$$

and let

$$\mathbb{E}_\mu(\varphi) := \int_X \varphi \, d\mu.$$

Theorem (Birkhoff)

Given a ergodic T-invariant probability μ and an observable function $\varphi : X \to \mathbb{R}$ with $\varphi \in L^1(\mu)$. Then

$$\lim_{n\to\infty} \mu \left\{ x \in X \ : \ \left| \frac{S_0^n \varphi(x) - n\,\mathbb{E}_\mu(\varphi)}{n} \right| \neq 0 \right\} = 0.$$

How does the Cesàro sum of φ distribute around the expectation?

Definition (Central Limit Theorem)

The measure μ satisfies the Central Limit Theorem with respect to a class of observables $\mathcal{H}$ if there exists $\sigma > 0$ such that, for every $\varphi \in \mathcal{H}$:

$$\lim_{n\to\infty} \mu \left\{ x \in X \; : \; \frac{S_0^n \varphi(x) - n\,\mathbb{E}_\mu(\varphi)}{\sqrt{n}} \leq t \right\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{t} e^{-u^2/2\sigma^2} du.$$

Question

How robust is the Central Limit Theorem? Does it still hold for :

- *approximations of the measure μ?*
- *incomplete readings of Cesàro sums (time)?*
- *approximate readings of Cesàro sums (space)?*

New methods of proof of the Central Limit theorem based on a specification property by periodic points (other reference points work as well).

To describe incomplete and approximate Cesàro sums introduce dynamical arrays.
Uniform case:

Definition (Dynamical Array)

For $l \in \mathbb{N}$ and $a_j \in \mathbb{N}$, $j = 1, \ldots, k_l$ consider intervals $I_j = [a_j, a_j + n_l]$ of uniform *length* n_l and uniform *gap size* $M_l = a_{j+1} - (a_j + n_l)$.

A Dynamical Array is a sequence of real valued functions defined as

$$S_{a_j}^{a_j+n_l}\varphi_l(x) := \sum_{i=a_j}^{a_j+n_l} \varphi_l\left(T^i x\right), \qquad j = 1, \ldots, k_l$$

where the $\varphi_l : X \to \mathbb{R}$.

One can also consider the more general (non-uniform) case where the length and the gap size are allowed to vary.

Denote

$$P_n := \{x \in X\colon\ T^n x = x\} \qquad \text{(periodic points)}$$

$$B^n_\varepsilon(x) := \{y \in X\colon\ d(T^k y, T^k x) < \varepsilon \quad k = 0, \dots, n\} \qquad \text{(Bowen ball)}$$

Definition (Global Specification)

A dynamical systems has global specification provided:
For every $\varepsilon > 0$ there exists $M(\varepsilon) \in \mathbb{N}$ such that:
for any $x_1, \dots, x_k \in X$ and $n \in \mathbb{N}$ and $M > M(\varepsilon)$, there exists $p \in P_{k(n+M)}$ with

$$T^{(i-1)(n+M)} p \in B^n_\varepsilon(x_i) \qquad i = 1, \dots, k.$$

Concatenation of k orbit pieces of length n can be shadowed by a single periodic orbit, provided sufficient time is allowed to migrate from one orbit to the next.

Specification (Local)

Orbits that are close enough at their initial and terminal time periods can by shadowed by periodic points.

Definition (Local Specification)

A dynamical systems has local specification provided:
For every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ and $N(\varepsilon) \in \mathbb{N}$ such that:
for any $x_1, ..., x_k \in X$ and $n > N(\varepsilon)$ with

$$d(T^n x_i, x_{i+1}) < \delta \qquad i = 1, \ldots k \quad \text{with} \quad x_{k+1} = x_1$$

there exists $p \in P_{kn}$ with

$$T^{(i-1)n} p \in B^n_\varepsilon(x_i) \qquad i = 1, ..., k.$$

For topologically mixing systems:

Local specification $\Longrightarrow$ Global specification

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For topologically mixing systems:

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Independence (of random variables) for (periodic) orbits

Definition (Global ε-independence)

Let $\varepsilon > 0$ and $k, n, M \in \mathbb{N}$. A subset $\mathcal{P} \subset P_{k(n+M)}$ is ε-independent if there exist:

- a subset $E \subset P_{n+M}$ which $(n, 3\varepsilon)$-spans $P_{k(n+M)}$
- a bijection $\pi : E^k \to \mathcal{P}$

such that for any $\underline{\mathbf{p}} = (p_1, ..., p_k) \in E^k$ and $1 \leqslant i \leqslant k$,

$$T^{(i-1)(n+M)}(\pi(\underline{\mathbf{p}})) \in B^n_\varepsilon(p_i).$$

Definition (Local ε-independence)

Let $\varepsilon > 0$ and $k, n \in \mathbb{N}$. Let $\mathcal{U}$ be a family of open sets and $A \in \bigvee_{i=0}^{k-1} T^{-in}\mathcal{U}$. A subset $\mathcal{P} \subset P_{kn}$ is locally ε-independent with respect to A if there exist

- $E_i \subset T^{(i-1)n}A$, $1 \leqslant i \leqslant k$
- a bijection π from $\prod_{i=1}^k E_i$ to $\mathcal{P}$

such that for any $\underline{x} = (x_1, ..., x_k) \in \prod_{i=1}^k E_i$ and $1 \leqslant i \leqslant k$.

$$T^{(i-1)n}(\pi(\underline{x})) \in B^n_\varepsilon(x_i).$$

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$$T^{(i-1)n}(\pi(\underline{\mathbf{x}})) \in B_\varepsilon^n(x_i).$$

Theorem A

Let $\nu_{\mathcal{P}_l}$ denote the equidistribution on a set $\mathcal{P}_l$ of periodic points .

Theorem (Main Theorem – global version)

Let $\{\epsilon_l > 0, k_l, n_l, M_l \in \mathbb{N}\}_{l\in\mathbb{N}}$ be sequences of numbers, with $\lim_{l\to\infty} k_l = \infty$. Consider a dynamical system (X,T) with global specification and a sequence of ε_l-independent sets $\mathcal{P}_l$. For observables φ_l satisfying

- *an oscillation condition*
- *a gap condition*

the Lindeberg condition with respect to the uniform measure $\nu_{\mathcal{P}_l}$ on $\mathcal{P}_l$ implies the central limit theorem:

$$\lim_{l\to\infty} \nu_{\mathcal{P}_l}\left(\left\{x \in X : \sum_{j=0}^{k_l(n_l+M_l)} \left(\varphi_l(T^j x) - \mathbb{E}_{\nu_{\mathcal{P}_l}}(\varphi_l)\right) \leqslant t\, s_l\right\}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du$$

where $$s_l^2 := \sum_{j=1}^{k} \sigma_{\mathcal{P}_l}^2 (S_{a_j}^{a_j+n_l} \varphi_l) \qquad \text{and} \qquad a_j = (j-1)(n_l + M_l).$$

The reverse holds true under a uniform oscillation condition.

Conditions

Oscillation condition

$$\lim_{l\to\infty} \frac{1}{s_l^i} \sum_{j=1}^{k_l} \int \left(\omega_{a_j}^{a_j+n_l}(\varphi_l, 4\epsilon_l, p)\right)^i d\nu_{\mathcal{P}_l}(p) = 0, \quad i = 1, 2,$$

where $\omega_m^n(\varphi, \epsilon, x) := \sup\left\{\left|S_m^n\varphi(x) - S_m^n\varphi(y)\right| : y \in B_\epsilon^n(T^m x)\right\}$

Gap condition

$$\lim_{l\to\infty} \frac{1}{s_l^2}\ \sigma^2_{\mathcal{P}_l}\left(\sum_{j=1}^{k_l} S_{a_j+n_l}^{a_{j+1}}\varphi_l\right) = 0.$$

Lindeberg condition

$$\lim_{l\to\infty} \frac{1}{s_l^2} \sum_{j=1}^{k_l} L_{\mathcal{P}_l}(S_{a_j}^{a_j+n_l}\varphi_l, \eta s_l) = 0, \quad \forall \eta > 0$$

where

$$L_P(\varphi, \eta) := \int (\varphi(z) - \mathbb{E}_P(\varphi))^2 \mathbb{1}_{\{|\varphi(z)-\mathbb{E}_P(\varphi)|>\eta\}}(z) d\nu_P.$$

The uniform oscillation condition

Uniform oscillation Condition

$$\lim_{l\to\infty} \frac{1}{\sigma^2_{\mathcal{P}_l}(S_0^{n_l}\varphi_l)} \int \left(\omega_0^{n_l}(\varphi_l, 2\epsilon_l, p)\right)^2 d\nu_{\mathcal{P}_l}(p) = 0$$

Theorem A holds for the uniform measures $\nu_{\mathcal{P}_l}$ on the ε_l independant sets.

But, accounting for multiplicity one also has:

Theorem

There exists a weighted distribution on the set of all periodic points for which a Lindeberg-type Central Limit Theorem as Theorem A holds.

Moreover, ε-independent (sub)-sets carry a rich structure. Indeed, for systems with a unique mesasure of maximal entropy μ_0

Theorem

Any sequence of uniform distributions on ε-independant sets converges to the measure of maximal entropy μ_0 (in the weak topology)

EXTRA CREDIT

For $\nu_{\mathcal{P}_l}$ typical periodic points, Birkhoff averages of full orbits approximate the expectation of Lipshitz observables.

Theorem (Theorem B)

Consider a positively expansive dynamical system (X, T) with global specification.

Let $\{\epsilon_l > 0, k_l, n_l, M_l \in \mathbb{N}\}_{l\in\mathbb{N}}$ be sequences of numbers with some mild conditions on the constants $N_l = k_l(n_l + M_l)$ and ε_l. Consider a sequence of ε_l-independent sets $\mathcal{P}_l \subset P_{k_l(n_l+M_l)}$.

For any Lipschitz function ψ and any $\eta > 0$ one has

$$\lim_{l\to\infty} \nu_{\mathcal{P}_l}\left(\left\{\left|\frac{1}{N_l}\sum_{j=0}^{N_l}\left(\psi(T^j x) - \mathbb{E}_{\nu_{\mathcal{P}_l}}(\psi)\right)\right| \leqslant k_l^{-\frac{1}{2}+\eta}\right\}\right) = 1.$$

Moreover, if (X, T) admits a unique measure of maximal entropy and

$$\sum_{l\in\mathbb{N}} \frac{1}{\sqrt{k_l}} \cdot \sup_{\|\psi\|_{\mathrm{Lip}}\leqslant 1} \frac{\mathbb{E}_{\mathcal{P}_l}(S^{n_l}\psi)^4}{\sigma^4_{\mathcal{P}_l}(S^{n_l}\psi)} < \infty,$$

then the uniform distributions over the orbit of random sequences of periodic points $p_l \in \mathcal{P}_l$ converge to the measure of maximal entropy.

EXTRA CREDIT

Theorem (Main Theorem – Local version)

Let $\{\epsilon_l > 0, k_l, n_l, M_l \in \mathbb{N}\}_{l\in\mathbb{N}}$ be sequences of numbers, with $\lim_{l\to\infty} k_l = \infty$.
Consider a dynamical system (X, T) with local specification and a sequence of locally ε_l-independent sets $\mathcal{P}_l$. For observables f_l satisfying

- *an oscillation condition*

the Lindeberg condition with respect to the uniform measure $\nu_{\mathcal{P}_l}$ on $\mathcal{P}_l$ holds

if and only if

the central limit theorem holds

$$\lim_{l\to\infty} \nu_{\mathcal{P}_l}\left(\left\{x \in X : \sum_{j=0}^{k_l(n_l+M_l)} \Big(\varphi_l(T^j x) - \mathbb{E}_{\nu_{\mathcal{P}_l}}(\varphi_l)\Big) \leqslant t\, s_l\right\}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du$$

and the array is asymptotically negligible, i.e.

$$\lim_{l\to\infty} \max_{1\leqslant j\leqslant k_l} \nu_{\mathcal{P}_l}\left(\left\{ \left|S_{a_j}^{a_j+n_l}\varphi_l - \mathbb{E}_{\nu_{\mathcal{P}_l}}(S_{a_j}^{a_j+n_l}\varphi_l)\right| \geqslant \eta s_l\right\}\right) = 0$$

EXTRA CREDIT

Theorem (Theorem D)

Let (X, T) be an expansive, topologically mixing dynamical system with the local specification property and a unique measure of maximal entropy. Then - with respect to the unique measure of maximal entropy μ- the class of wildly oscillating functions in $L^3(\mu)$ satisfying a condition on the moments and with integrable local variance belongs to the partial domain of attraction of a mixed normal distribution, i.e. a subsequence of properly centered and normed partial sums converges weakly to a mixed normal distribution.

Thank you!