Constant Slope Models and Perturbation

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Joint work with Michal Malek

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Papaver rhoeas – Field poppy – Mak polny – Vlčí mak

Michal Misiurewicz – a student's perspective

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Spaces of transitive interval maps

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The area of dynamical systems where one investigates dynamical properties that can be described in topological terms is called *topological dynamics*. Investigating the topological properties of spaces of maps that can be described in dynamical terms is in a sense the opposite idea. Therefore we propose to call this area *dynamical topology*. It is on the boundary between dynamical systems and topology, but, in our opinion, much closer to dynamical systems, because most of the tools that can be used there are from dynamics.

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Definitions:

Our space of maps:

$$\mathcal{T}_{PM} = \{f : [0,1] \to [0,1]\,;\, f \text{ is continuous, transitive, and piecewise monotone}\}$$

We use the metric of uniform convergence:

$$d(f,g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|$$

Piecewise monotone means finitely many critical points:

$$\text{Crit}(f) = \{0,1\} \bigcup \{x\,;\, f \text{ is not monotone on any neighborhood of } x\}$$

The modality of f is $\#\,\text{Crit}(f) \cap (0,1)$. We usually make perturbations preserving modality, i.e. in a subspace:

$\mathcal{T}_m \subset \mathcal{T}_{PM}$ – the subspace of maps of modality m.

Constant Slope:

$f \in \mathcal{T}_{PM}$ has <u>constant slope</u> λ if $|f'(x)| = \lambda$ for $x \notin \text{Crit}(f)$.

A <u>constant slope model</u> for $f \in \mathcal{T}_{PM}$ is a conjugate map $\tilde{f} = \varphi^{-1} \circ f \circ \varphi$ of constant slope, where the conjugating homeomorphism φ is orientation-preserving (increasing).

$$\begin{array}{ccc} [0,1] & \xrightarrow{\tilde{f} \text{ - constant slope}} & [0,1] \\ \varphi \big\downarrow & & \big\downarrow \varphi \in \text{Homeo}_+([0,1]) \\ [0,1] & \xrightarrow[f \text{ - transitive, p.m.}]{} & [0,1] \end{array}$$

Theorem *Parry (1966), Alseda and Misiurewicz (2015)*
Each map $f \in \mathcal{T}_{PM}$ has a unique constant slope model.
The conjugating homeomorphism is likewise unique.
The constant slope is $\lambda = \exp h(f)$.

Objects of interest:

$$\begin{aligned} \Phi &: \mathcal{T}_{PM} \to \mathcal{T}_{PM}, & \Phi(f) &= \text{the constant slope model for } f \\ \Psi &: \mathcal{T}_{PM} \to \text{Homeo}_+([0,1]), & \Psi(f) &= \text{the conjugating homeomorphism} \\ h &: \mathcal{T}_{PM} \to \mathbb{R}, & h(f) &= \text{the topological entropy of } f \end{aligned}$$

Question: How does Φ behave?

Is Φ continuous?
In other words, as $g \rightrightarrows f$, does $\Phi(g) \rightrightarrows \Phi(f)$?

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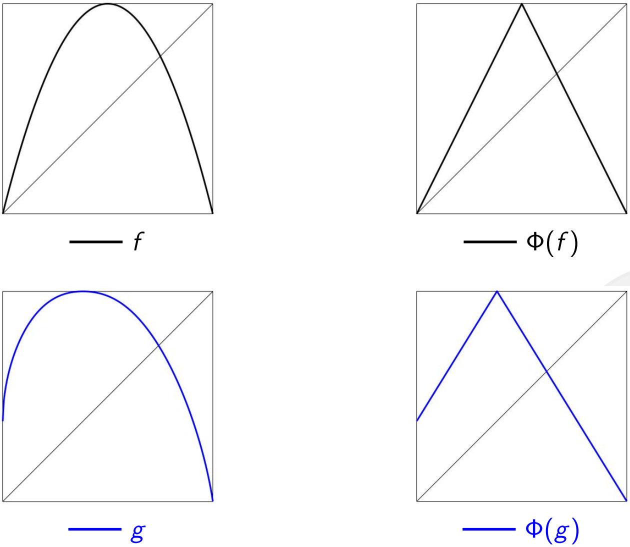

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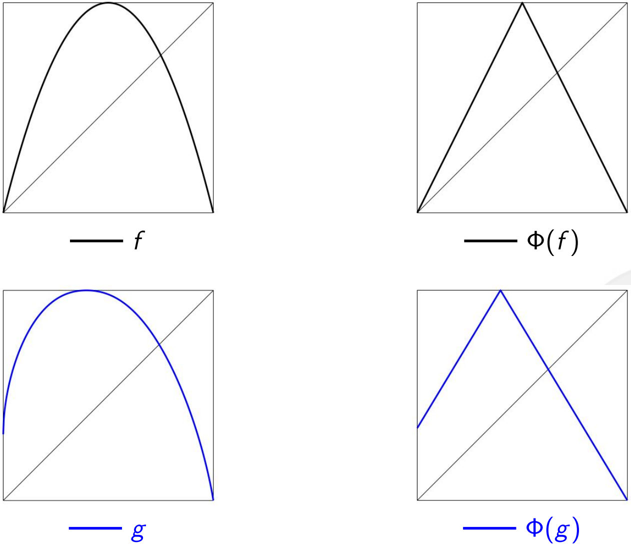

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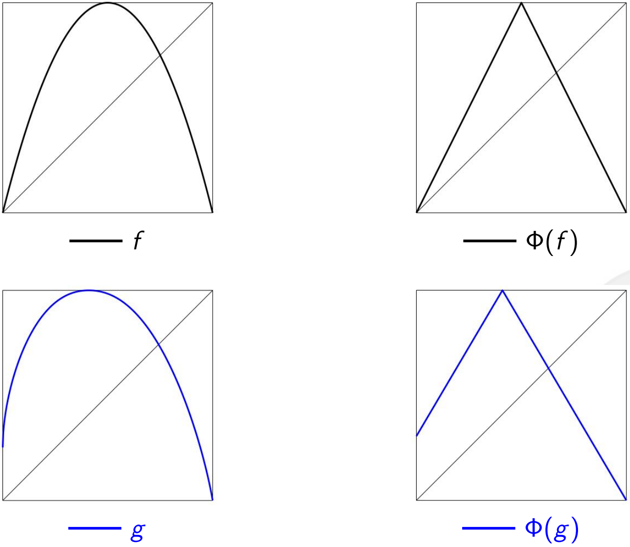

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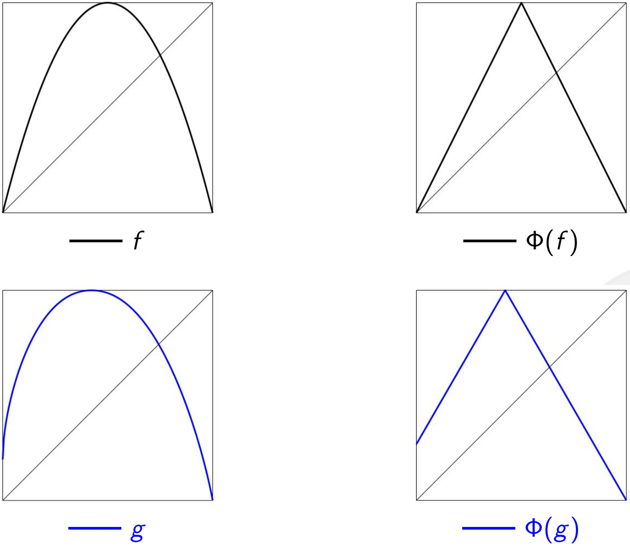

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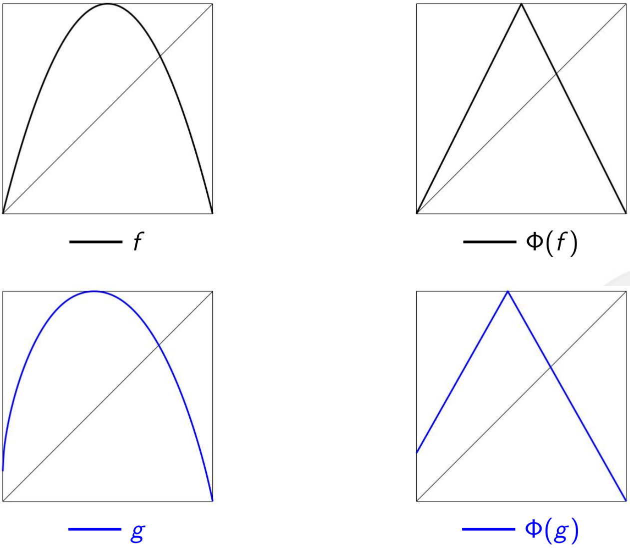

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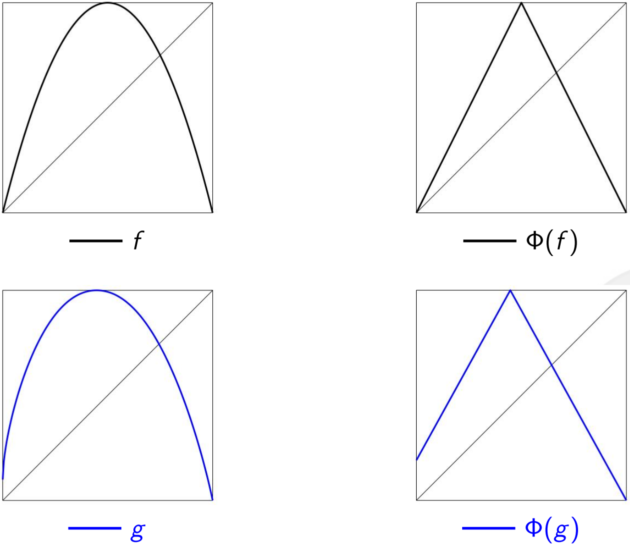

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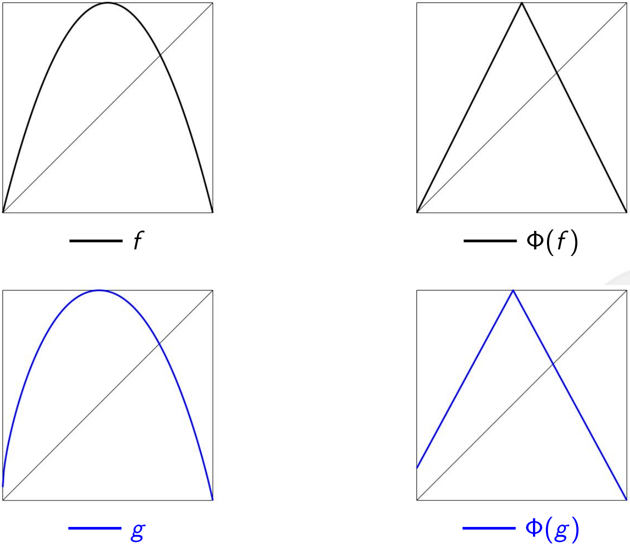

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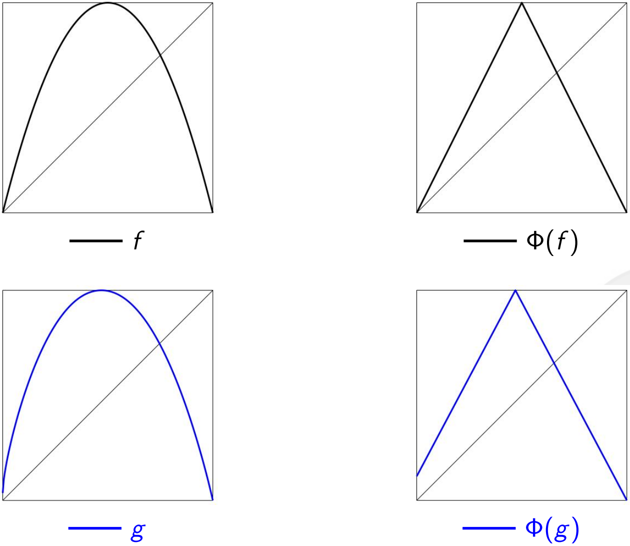

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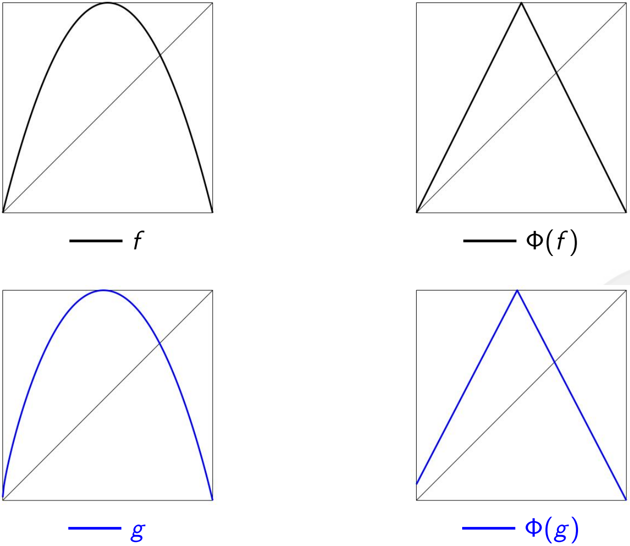

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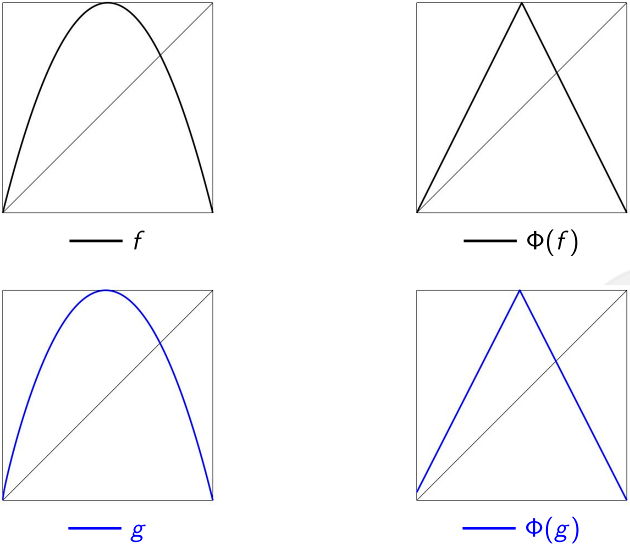

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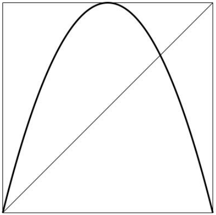
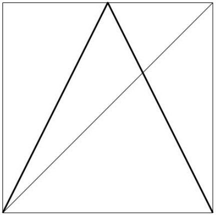
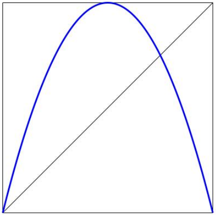


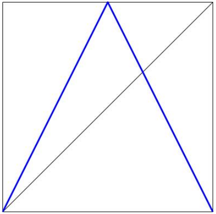
— f



— $\Phi(f)$



— limit



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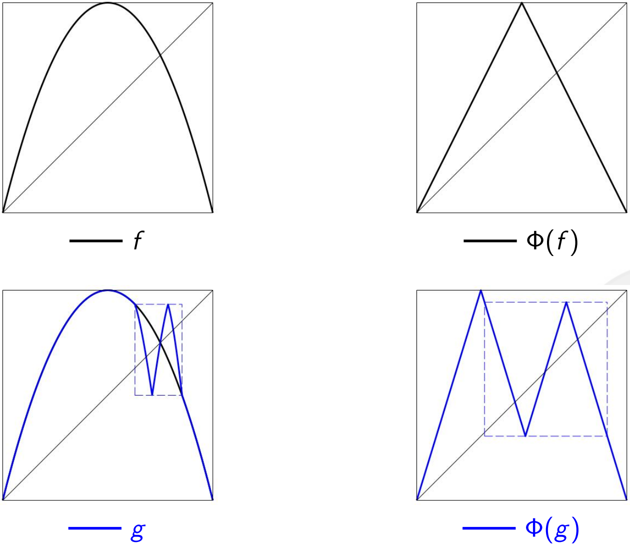

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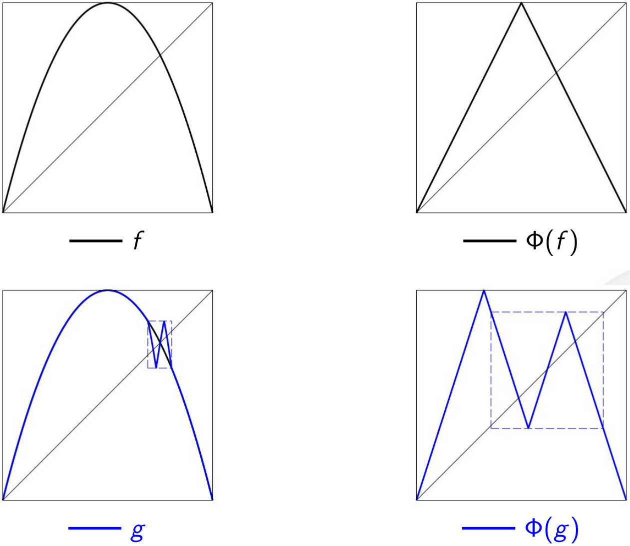

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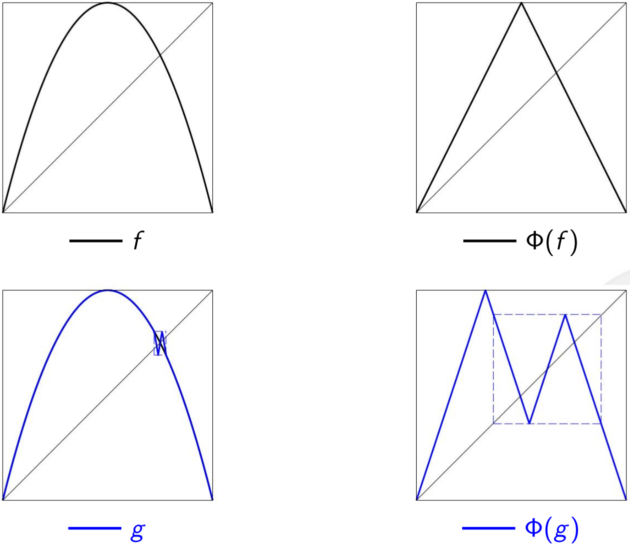

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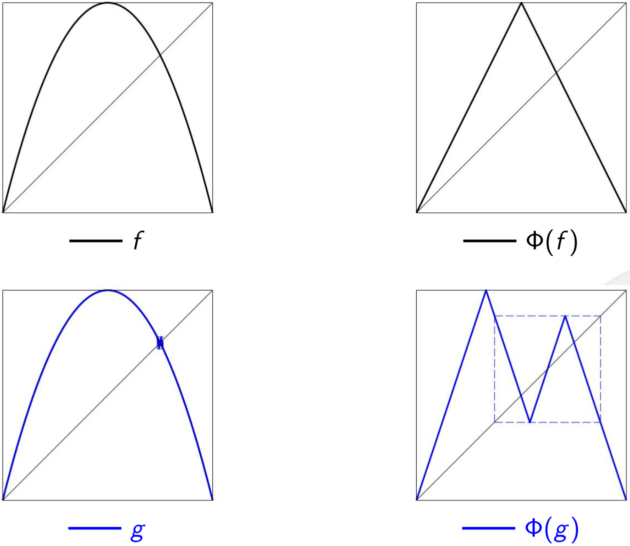

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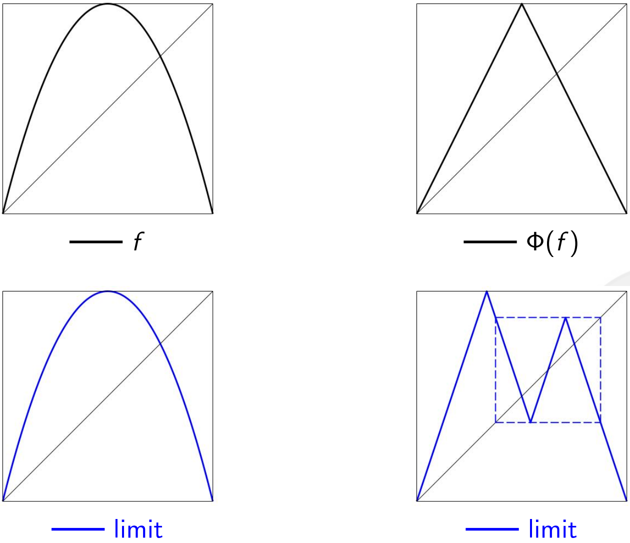

Conclusion: Φ is not continuous.

There are two ways to get $g_n \rightrightarrows f$ but $\Phi(g_n) \not\rightrightarrows \Phi(f)$.

Obstacle (1): Jumps in entropy.
Obstacle (2): Jumps in modality.

Restricting to maps of a fixed modality m,

$$\Phi_m : \mathcal{T}_m \to \mathcal{T}_m, \quad \Phi_m(f) = \text{the constant slope model for } f$$

Revised question: Is Φ_m continuous?

Theorem *Misiurewicz (2001)*
$h : \mathcal{T}_m \to \mathbb{R}$ has discontinuity points if and only if $m \geq 5$.

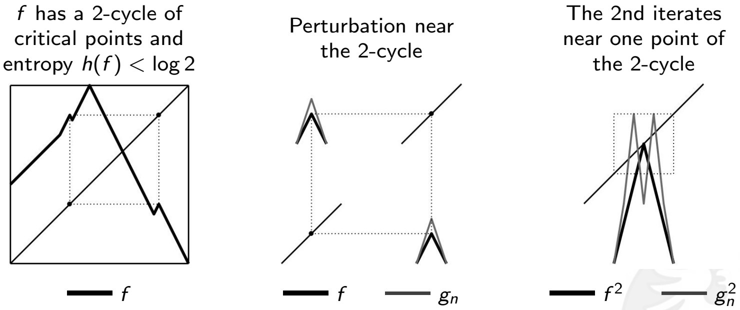

Figure: A discontinuity point of $h : \mathcal{T}_5 \to \mathbb{R}$.

Theorem *Misiurewicz, Alseda (2015)*
If $f \in \mathcal{T}_m$ is a discontinuity point of h, then it is a discontinuity point of Φ_m.

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SEMICONJUGACY TO A MAP OF A CONSTANT SLOPE

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Corollary 2. *If $f \in \mathcal{T}_n$ is a point of discontinuity of the topological entropy as a function from $\mathcal{T}_n$ to $\mathbb{R}$, then f is also a point of discontinuity of the operator Φ_n.*

We conjecture that the operator Φ_n is continuous at every point of continuity of the topological entropy on $\mathcal{T}_n$.

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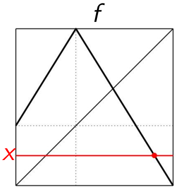
We confirm that conjecture:

Theorem 1
If $g_n \rightrightarrows f$ in $\mathcal{T}_m$ and $h(g_n) \to h(f)$, then $\Phi(g_n) \rightrightarrows \Phi(f)$.

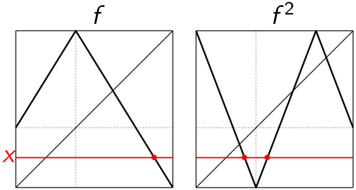
Corollary Φ_m is continuous for $m \leq 4$ and discontinuous for $m \geq 5$.

Counting preimages



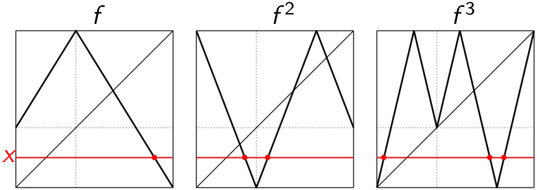
n	1
$\#f^{-n}(x)$	1

Counting preimages



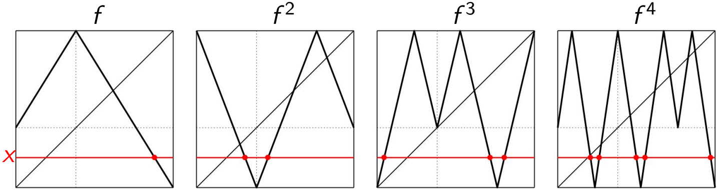
n	1	2
$\#f^{-n}(x)$	1	2

Counting preimages



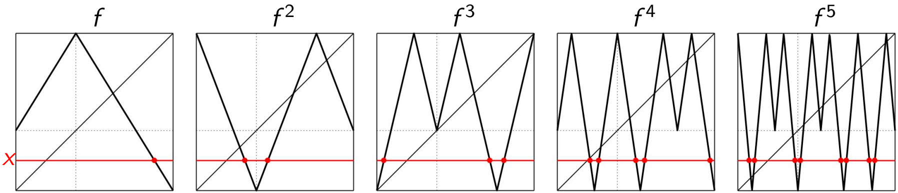
n	1	2	3
$\#f^{-n}(x)$	1	2	3

Counting preimages



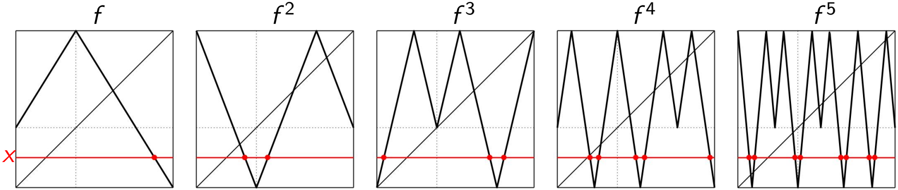
n	1	2	3	4
$\#f^{-n}(x)$	1	2	3	5

Counting preimages



n	1	2	3	4	5	$\cdots$
$\#f^{-n}(x)$	1	2	3	5	8	$\cdots$

Counting preimages



n	1	2	3	4	5	$\cdots$
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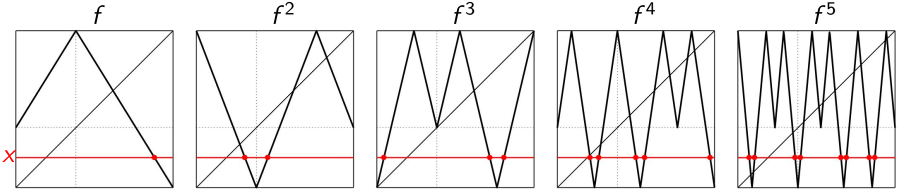

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Well-known fact

Let $f \in \mathcal{T}_{PM}$, $x \in [0,1]$. Then preimage counts grow like the entropy:

$$\lim_{n\to\infty} \frac{1}{n} \log \#f^{-n}(x) = h(f).$$

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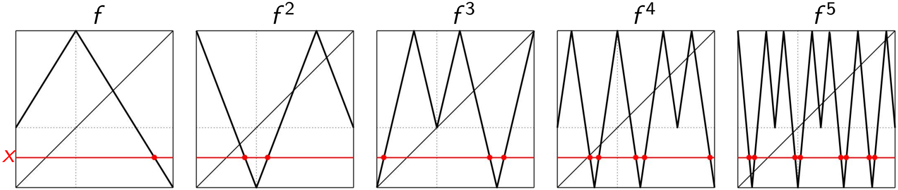

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$$\lim_{n\to\infty} \frac{1}{n} \log \#f^{-n}(x) = h(f).$$

Proof

($\leq$) The number of laps of f^n grows like the entropy.
($\geq$) By transitivity, x has a preimage in each horseshoe, and entropy is given by horseshoes.

Counting preimages

Question What about the “subexponential term?”

$$\lim_{n\to\infty} \frac{\#f^{-n}(x)}{e^{nh(f)}} = ??$$

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- *Lasota and Yorke (1973):* Convergence a.e. w.r.t. the measure of constant Jacobian.
- A transitive subshift of finite type is called *positive recurrent* iff the analogous limit is not zero.

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- *Lasota and Yorke (1973):* Convergence a.e. w.r.t. the measure of constant Jacobian.
- A transitive subshift of finite type is called *positive recurrent* iff the analogous limit is not zero.

Theorem 2

Let $f \in \mathcal{T}_{PM}$, $x \in [0,1]$. When we count preimages, the subexponential term does not converge to zero:

$$\limsup_{n\to\infty} \frac{\# f^{-n}(x)}{e^{nh(f)}} > 0.$$

Interpretation

Transitive piecewise monotone interval maps are "positive recurrent."

Theorem 1 If $g_n \rightrightarrows f$ in $\mathcal{T}_m$ and $h(g_n) \to h(f)$, then $\Phi(g_n) \rightrightarrows \Phi(f)$.

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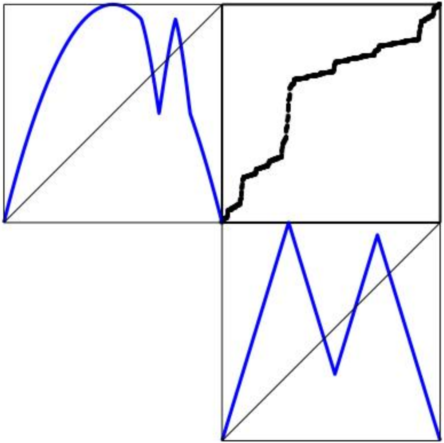
Question: What happens to the conjugating homeomorphism?



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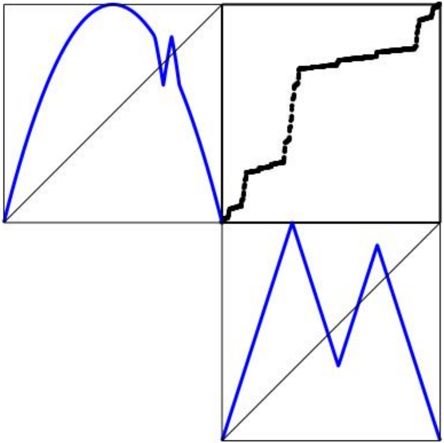

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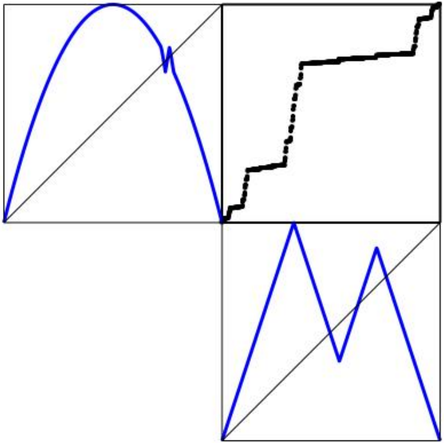

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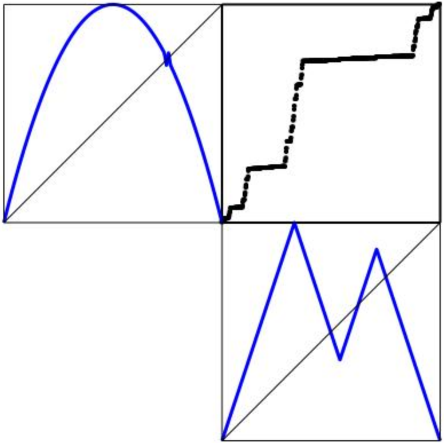

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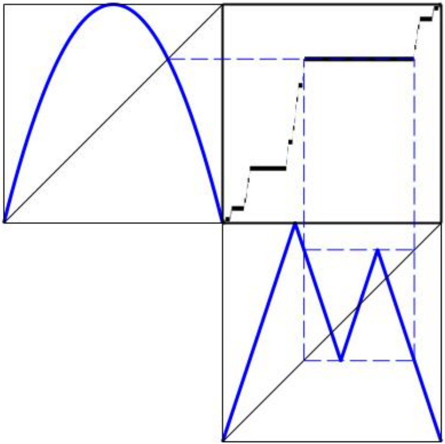

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For piecewise monotone interval maps:

transitive $\Leftarrow$ mixing $\Leftrightarrow$ locally eventually onto

If f is transitive but not mixing, then it has a unique fixed point e, it interchanges $[0,e]$ with $[e,1]$, and the restricted maps $f^2|_{[0,e]}$, $f^2|_{[e,1]}$ are both mixing.

In this presentation, we assume that f is mixing.

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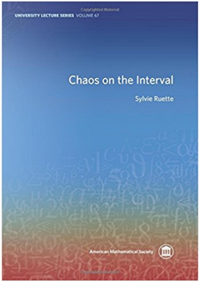
Fact *"Accessible endpoints"*

$$f^2((0,1)) = [0,1]$$



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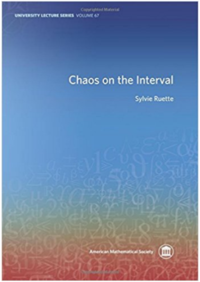

Lemma *"Equi-accessible endpoints"*

$$\exists \rho, \zeta > 0 : (g \in \mathcal{T}_m \wedge d(f,g) < \zeta) \implies g^2([\rho, 1-\rho]) = [0,1]$$

Theorem 1 If $g_n \rightrightarrows f$ in $\mathcal{T}_m$ and $h(g_n) \to h(f)$, then $\Phi(g_n) \rightrightarrows \Phi(f)$.

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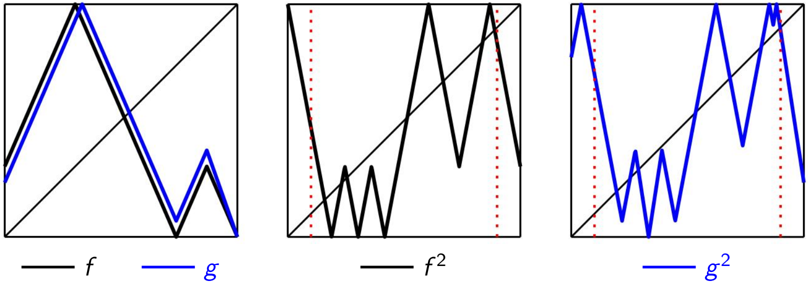
$$f^2((0,1)) = [0,1]$$

Lemma *"Equi-accessible endpoints"*

$$\exists \rho, \zeta > 0 : (g \in \mathcal{T}_m \wedge d(f,g) < \zeta) \implies g^2([\rho, 1-\rho]) = [0,1]$$



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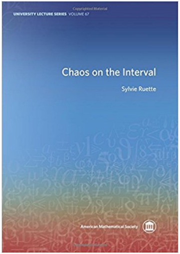
Fact *"Uniformly l.e.o."*

$\forall \epsilon > 0, \ \exists K :$

$y - x > \epsilon \implies f^K([x, y]) = [0, 1].$



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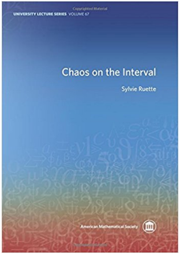
Lemma *"Equi-uniformly l.e.o."*

$\forall \epsilon > 0, \ \exists K, N, \ \forall n \geq N :$

$y - x > \epsilon \implies g_n^{K+2}([x,y]) = [0,1].$



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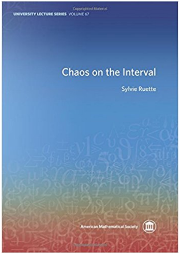

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Proof

Choose N such that for $n \geq N$, $d(f^K, g_n^K) < \rho$ and $d(f^2, g_n^2) < \zeta$. Then

$$g_n^2 \circ g_n^K([x,y]) \supseteq g_n^2([\rho, 1-\rho]) = [0,1].$$

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Proposition

(ψ_n) is an equicontinuous family.

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Let L be a common Lipschitz constant for the maps $\Phi(g_n)$.

$$g_n^{K+2}([x,y]) = [0,1] \implies (\Phi(g_n))^{K+2}\left([\psi_n^{-1}x, \psi_n^{-1}y]\right) = [0,1]$$
$$\implies \psi_n^{-1}y - \psi_n^{-1}x \geq L^{-k-2}.$$

Taking $\delta = L^{-k-2}$, we have shown that

$$\forall \epsilon > 0,\ \exists N, \delta,\ \forall n \geq N: \quad y - x > \epsilon \implies \psi_n^{-1}y - \psi_n^{-1}x > \delta.$$

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Corollary *Arzela-Ascoli:*
Every subsequence of (ψ_n) has a further subsequence which converges uniformly.

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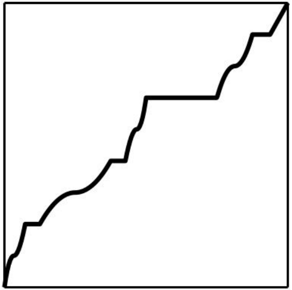
Properties of a subsequential limit ψ of the sequence (ψ_n):



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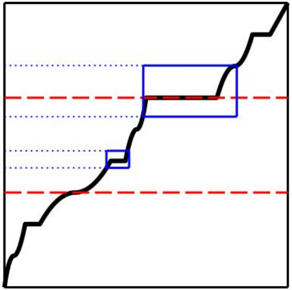

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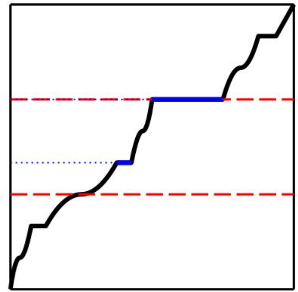

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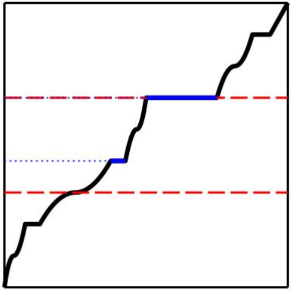

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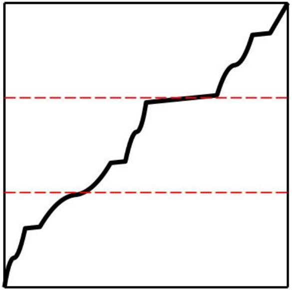

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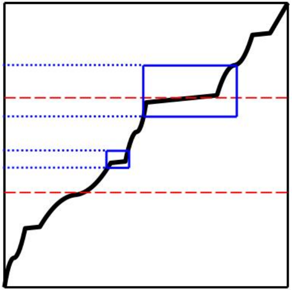

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Put $\tilde{f} = \psi^{-1} \circ f \circ \psi$.
If $[x, x'] \cap \mathrm{Crit}(\tilde{f}) = \emptyset$, then $|\tilde{f}(x') - \tilde{f}(x)| = \lambda |x' - x|$.

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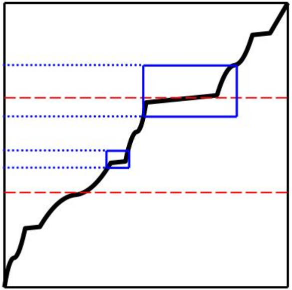

Concluding Arguments

- By uniqueness of constant slope models, $\tilde{f} = \Phi(f)$.
- By uniqueness of the conjugating homeomorphism, $\psi = \varphi$.
- Therefore $\psi_n \rightrightarrows \varphi$.
- Therefore $\Phi(g_n) \rightrightarrows \Phi(f)$.

Put $\tilde{f} = \psi^{-1} \circ f \circ \psi$.
If $[x, x'] \cap \mathrm{Crit}(\tilde{f}) = \emptyset$, then $|\tilde{f}(x') - \tilde{f}(x)| = \lambda |x' - x|$.

Thanks so much for your attention :)