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Transitivity and mixing for expanding Lorenz maps

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Kwiaty z Alp

Szarotka alpejska

Leontopodium nivale, formerly: **Leontopodium alpinum**

German: Edelweiß (Switzerland: Edelweiss),
English, French, Spanish, Dutch, Swedish, Norvegian (both), Danish,
Basque, Galician, Albanian, Irish, Turkish: Edelweiss,
Portugues: Edelvaisse, Greek: Εντελβάις,
Russian: Эдельвейс, Bulgarian: Еделвайс,
Italian: Stella alpina, Catalan: Flor de neu, Slovenian: Planika,
Slovak, Czech: Plesnivec, Croatian, Bosnian, Serbian: Runolist,
Ukrainian: Білотка, Romanian: Floarea reginei, Hungarian: Havasi gyopár,
Finnish: Alppitähti, Lithuanian: Liūtpėdė, Estonian: Jänesekäpp

Most of the results presented in this talk were obtained together with **Piotr Oprocha** and **Paweł Potorski**, some results were obtained together with **Angela Berger** (**Stachelberger**).
Let $f : [0,1] \to [0,2]$ be a continuous strictly increasing function which is differentiable on $(0,1) \setminus F$ where F is a finite set. Define $\inf f' := \inf_{x \in (0,1) \setminus F} f'(x)$, and assume that $\inf f' > 1$. Hence there exists a unique $c \in (0,1)$ with $f(c) = 1$. Now define

$$T_f x := f(x) \pmod 1 = f(x) - \lfloor f(x) \rfloor \ ,$$

where $\lfloor y \rfloor$ is the largest integer smaller or equal to y. One calls a map T_f of this form an expanding Lorenz map. Then T_f is a piecewise monotonic map but it has a discontinuity at c. Moreover, $\lim_{x \to c^-} T_f x = 1$ and $\lim_{x \to c^+} T_f x = 0$. Using a standard doubling points construction one can apply all usual definitions of dynamical systems also to Lorenz maps.

A map T_f is called *topologically transitive* if there exists an $x \in [0,1]$ such that the ω-limit set (the set of all limit points of $({T_f}^n x)_{n \in \mathbb{N}}$) equals $[0,1]$. It is equivalent to the property that for any nonempty open $U, V \subseteq [0,1]$ there exists an $n \in \mathbb{N}$ such that ${T_f}^n U \cap V \neq \emptyset$. For expanding Lorenz maps with $\inf f' \geq \sqrt[3]{2}$ we get the following result.

Theorem 1

Set $\beta := \inf f'$ and assume that one of the following properties is satisfied.

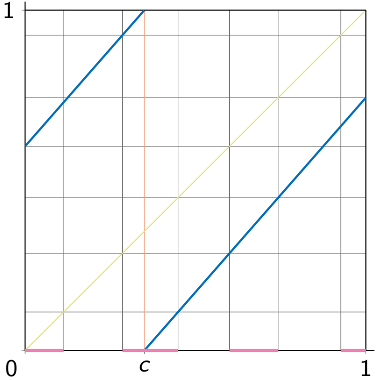
(1) $\beta \geq \sqrt{2}$.

(2) $\beta \geq \sqrt[3]{2}$ *and* $f(0) \geq \frac{1}{\beta+1}$.

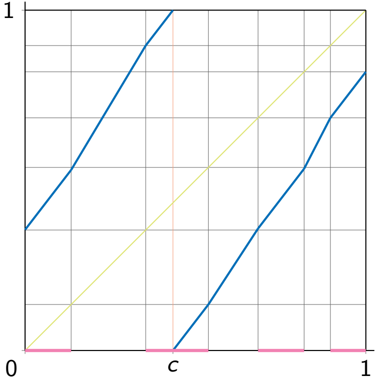
(3) $\beta \geq \sqrt[3]{2}$ *and* $f(1) \leq 2 - \frac{1}{\beta+1}$.

Then T_f is topologically transitive.

Define $f(x) := \frac{8}{7}x + \frac{3}{5}$. Here $\beta = \frac{8}{7} < \sqrt[3]{2}$ and $f(0) \geq \frac{1}{\beta+1}$. One can see the graph of T_f in the figure below. As the pink set is T_f-invariant the map T_f is not topologically transitive.



Consider the map whose graph is shown in the figure below. We have $\beta = \inf f' = \frac{13}{10} > \sqrt[3]{2}$ and $f(0) < \frac{1}{\beta+1}$ in this case. Since the pink set is T_f-invariant the map T_f is not topologically transitive.



Next we call T_f *topologically mixing* if for every nonempty open $U, V \subseteq [0,1]$ there exists an $N \in \mathbb{N}$ such that ${T_f}^n U \cap V \neq \emptyset$ for all $n \geq N$. This obviously implies topological transitivity. Our next result states that in the situation of Theorem 1 in only three cases T_f is not mixing.

Theorem 2

Set $\beta := \inf f'$ and assume that one of the following properties is satisfied.

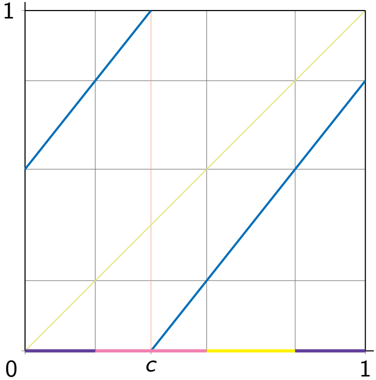
(1) $\beta \geq \sqrt{2}$ *and* $f \neq \sqrt{2}x + 1 - \frac{1}{\sqrt{2}}$.

(2) $\beta \geq \sqrt[3]{2}$, $f(0) \geq \frac{1}{\beta+1}$ *and* $f \neq \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$.

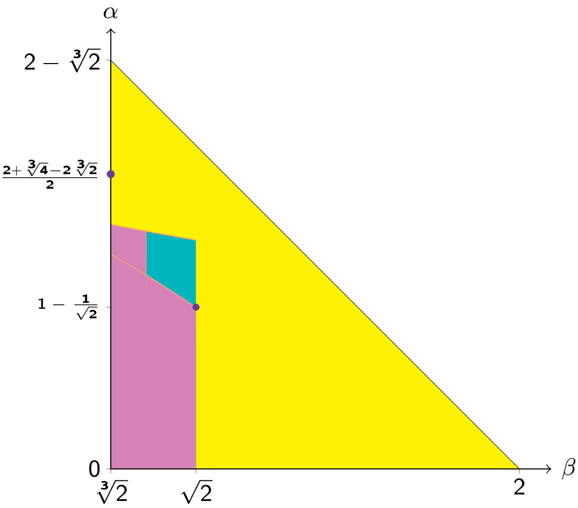
(3) $\beta \geq \sqrt[3]{2}$, $f(1) \leq 2 - \frac{1}{\beta+1}$ *and* $f \neq \sqrt[3]{2}x + \frac{2-\sqrt[3]{4}}{2}$.

Then T_f is topologically mixing.

For $f(x) := \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ the graph of T_f is shown in the figure below. In this example one can reach in one step only a violet interval from the pink interval, the yellow interval from one of the violet intervals, and the pink interval from the yellow interval. Hence T_f is not mixing.



By Theorem 1 and Theorem 2 we can draw the region of (β, α) with $\sqrt[3]{2} \leq \beta \leq 2$ such that $\inf f' \geq \beta$ and $f(0) \geq \alpha$ implies that T_f is topologically transitive.



Assuming that $f(x) = \beta x + \alpha$ we can obtain the following better result.

Theorem 3

Suppose that $f(x) = \beta x + \alpha$ *with* $\sqrt[3]{2} \leq \beta \leq 2$ *and* $0 \leq \alpha \leq 2 - \beta$. *Then* T_f *is topologically transitive if and only if*

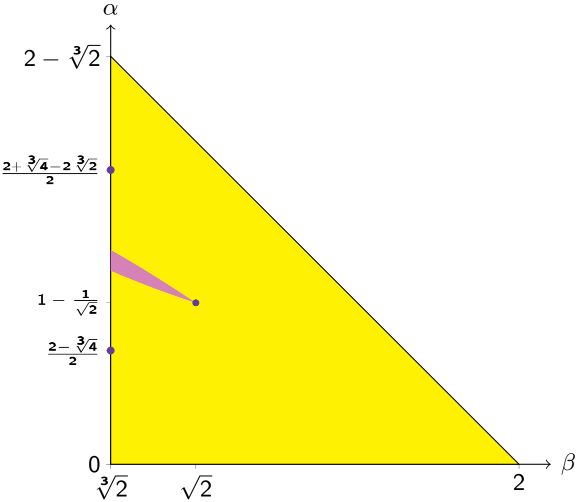
(1) $\beta \geq \sqrt{2}$, *or*

(2) $\sqrt[3]{2} \leq \beta \leq \sqrt{2}$ *and* $0 \leq \alpha < \frac{1}{\beta^2+\beta}$, *or*

(3) $\sqrt[3]{2} \leq \beta \leq \sqrt{2}$ *and* $2 - \beta - \frac{1}{\beta^2+\beta} < \alpha \leq 2 - \beta$.

Furthermore T_f *is topologically mixing if and only if* (1) *or* (2) *or* (3) *above is satisfied and*
$f \notin \left\{ \sqrt{2}x + 1 - \frac{1}{\sqrt{2}}, \sqrt[3]{2}x + \frac{2-\sqrt[3]{4}}{2}, \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2} \right\}$.

Using Theorem 3 we can draw the region of (β, α) with $\sqrt[3]{2} \leq \beta \leq 2$ such that $\beta x + \alpha \pmod 1$ is topologically transitive.



Trying to generalize this result leads to the notion of $n(k)$-cycles introduced by Palmer in 1979. If $\{z_0, z_1, \dots, z_{n-1}\}$ is a periodic orbit of period n and

$$z_0 < z_1 < \cdots < z_{n-k-1} < c < z_{n-k} < \cdots < z_{n-1}\,,$$

then it is called an *$n(k)$-cycle*. Moreover, an $n(k)$-cycle is called a *primary $n(k)$-cycle* if

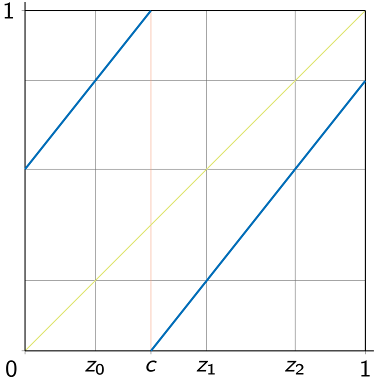
(1) $T_f z_j = z_{j+k \pmod{n}}$ for every $j \in \{0, 1, \dots, n-1\}$,

(2) k and n are coprime,

(3) $z_{k-1} \le T_f 0$ and $z_k \ge T_f 1$.

Paul Glendinning claimed in 1990 that an expanding Lorenz map with a primary $n(k)$-cycle cannot be topologically transitive. However, this is not true.

Consider the example T_f for $f(x) := \sqrt[3]{2}x + \frac{2+\sqrt[3]{4}-2\sqrt[3]{2}}{2}$ from above. Setting $z_0 := {T_f}^2 0$, $z_1 := T_f 0$ and $z_2 := {T_f}^3 0 = T_f 1$ we get a primary 3(2)-cycle (see the figure below). On the other hand T_f is topologically transitive by Theorem 1, but not topologically mixing by Theorem 2.



Investigating how transitivity and mixing are related to primary $n(k)$-cycles we obtain the following results.

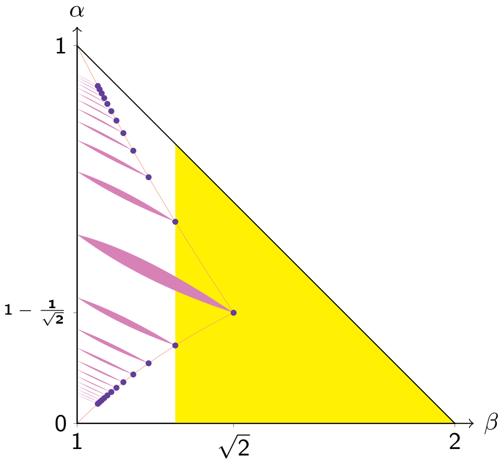
Theorem 4

Assume that T_f is an expanding Lorenz map having a primary $n(k)$-cycle $z_0 < z_1 < \cdots < z_{n-1}$. If $T_f 0 = z_{k-1}$ and $T_f 1 = z_k$ then T_f is topologically transitive but not topologically mixing.

Theorem 5

Suppose that T_f is an expanding Lorenz map having a primary $n(k)$-cycle $z_0 < z_1 < \cdots < z_{n-1}$. Moreover, assume that $T_f 0 > z_{k-1}$ or $T_f 1 < z_k$. Then T_f is not topologically transitive.

Next we draw a region of (β, α) where $\beta x + \alpha \pmod 1$ is not topologically transitive. For this region primary $n(1)$-cycles and primary $n(n-1)$-cycles are considered.



We recall some definitions essentially given by Paul Glendinning in 1990.

The map T_f is called *locally eventually onto* if for every nonempty open $U \subseteq [0,1]$ there exist open intervals $U_1, U_2 \subseteq U$ and $n_1, n_2 \in \mathbb{N}$ such that ${T_f}^{n_1}$ maps U_1 homeomorphically to $(0, c)$ and ${T_f}^{n_2}$ maps U_2 homeomorphically to $(c, 1)$.

One calls T_f *renormalizable* if there are $0 \leq u_1 < c < u_2 \leq 1$ and $l, r \in \mathbb{N}$ with $l + r \geq 3$ such that

(1) ${T_f}^l$ is continuous on (u_1, c),

(2) ${T_f}^r$ is continuous on (c, u_2),

(3) $\lim_{x \to c^-} {T_f}^l x = u_2$ and $\lim_{x \to c^+} {T_f}^r x = u_1$.

Because of some exceptional cases one defines T_f to be *special trivial renormalizable* if $T_f 0 = 0$ or $T_f 1 = 1$ or $T_f 0 = c$ or $T_f 1 = c$. Note that a special trivial renormalizable Lorenz map need not be renormalizable.

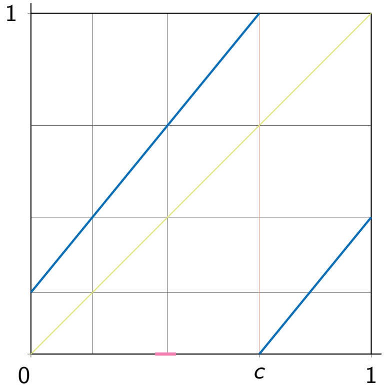
Every locally eventually onto Lorenz map is topologically mixing. It was widely believed that in the case it is not special trivial renormalizable T_f is locally eventually onto if and only if it is not renormalizable.

Unfortunately this is not true. Let β be the largest zero of the polynomial $x^4 - x - 1$, set $\alpha := 1 - \frac{1}{\beta}$ and define $f(x) := \beta x + \alpha$. We obtain that $c = \frac{1}{\beta^2}$ and

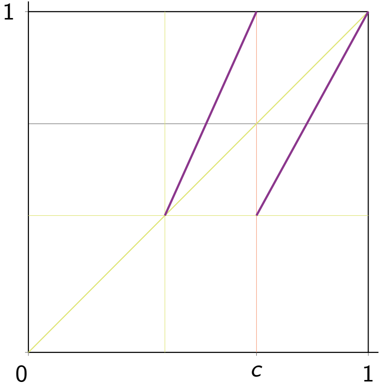
$$\beta \approx 1.2207440846057594753616853491088319144324890862486 4\,,$$
$$\alpha \approx 0.1808274866038355603004288116575729596515021674462 9\,,$$
$$c \approx 0.6710436067037892084168156540361997025527444747711 8\,.$$

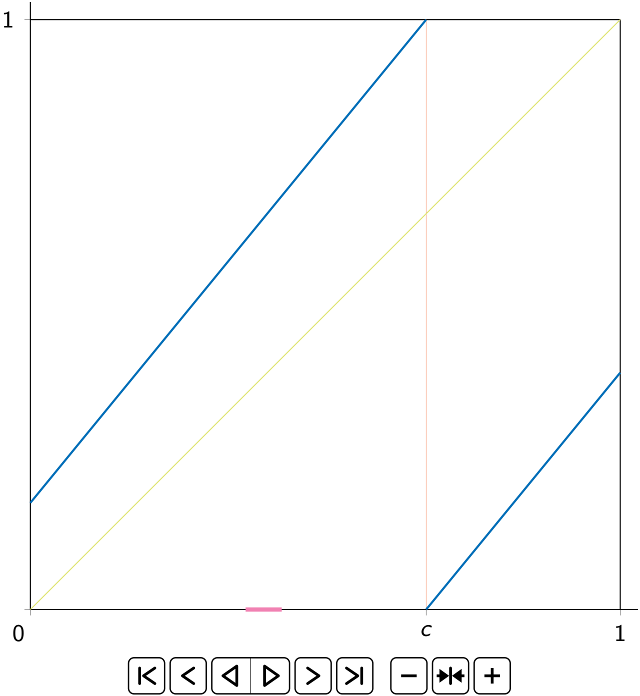
Furthermore we have ${T_f}^3 0 = {T_f}^2 1 = c$.

The graph of the map T_f is shown in the figure below. It can be shown that ${T_f}^{12}$ maps the pink interval homeomorphically to $(0, c)$, and this can be used to prove that T_f is locally eventually onto.



Set $u_1 := {T_f}^2 0$, $u_2 := 1$, $l := 4$ and $r := 3$. As shown in the figure below the map T_f is renormalizable.




1
0
c
1

Because of this example T_f may be locally eventually onto and renormalizable, even if it is not special trivial renormalizable.

One calls the map T_f *strongly locally eventually onto* if for every nonempty open $U \subseteq [0,1]$ there exist open intervals $U_1, U_2 \subseteq U$ and $n_1, n_2 \in \mathbb{N}$ such that ${T_f}^{n_1}$ maps U_1 homeomorphically to $(0, c)$, ${T_f}^{n_2}$ maps U_2 homeomorphically to $(c, 1)$, $T_f, {T_f}^2, \ldots, {T_f}^{n_1}$ are continuous on U_1 and $T_f, {T_f}^2, \ldots, {T_f}^{n_2}$ are continuous on U_2.

We obtain then the following result.

Theorem 6

For Lorenz maps the following conditions are equivalent.

(1) *The map T_f is strongly locally eventually onto.*

(2) *The map T_f is not renormalizable or T_f is special trivial renormalizable.*

Thank you very much

Dziękuję bardzo

Michale,
wszystkiego najlepszego
z okazji urodzin!