Spiders' webs in the punctured plane

David Martí-Pete
Institute of Mathematics of
the Polish Academy of Sciences

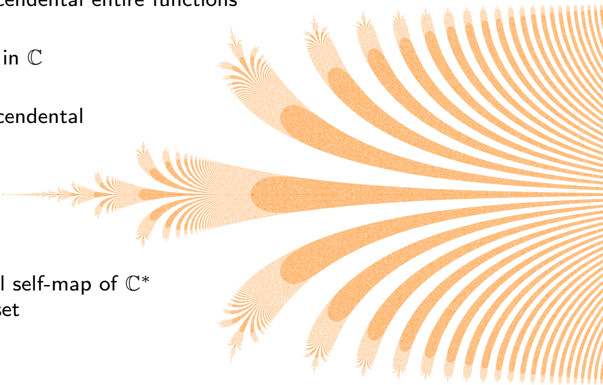
– joint work with Vasso Evdoridou and Dave Sixsmith –

Conference on Dynamical Systems
Celebrating Michał Misiurewicz's 70th Birthday
Jagiellonian University, Kraków (Poland)
June 10, 2019

Hortensia (*Hydrangea macrophylla*)

Sketch of the talk

1. The escaping set of transcendental entire functions
2. Definition of spider's web in $\mathbb{C}$
3. The escaping set of transcendental self-maps of $\mathbb{C}^*$
4. Adapting the notion of spider's web to $\mathbb{C}^*$
5. Example of transcendental self-map of $\mathbb{C}^*$ with connected escaping set
6. Sketch of the proof



Preprint: arXiv:1901.05276

The escaping set of transcendental entire functions

The **escaping set** of a transcendental entire function f,

$$I(f) := \{z \in \mathbb{C} \ : \ f^n(z) \to \infty \text{ as } n \to \infty\}$$

was introduced by Eremenko in 1989.

Theorem (Eremenko, 1989)

Let f be a transcendental entire function. Then,

I1. $I(f) \cap J(f) \neq \emptyset$;

I2. $J(f) = \partial I(f)$;

I3. *all the components of $\overline{I(f)}$ are unbounded.*

- **Eremenko's conjecture**: all the components of $I(f)$ are unbounded.
- **Strong Eremenko's conjecture**: every escaping point can be joined with ∞ by a curve in $I(f)$.

[Ere89] A. Eremenko, *On the iteration of entire functions*, Dynamical Systems and Ergodic Theory, Banach Center Publ. 23 (1989), 339-345.

In [R^3S11], a partial positive answer to the strong Eremenko's conjecture was given:

Theorem (Rottenfußer, Ruckert, Rempe and Schleicher, 2011)

Let $f \in \mathcal{B}$ be a function of finite order, or more generally a finite composition of such functions. Then every point $z \in I(f)$ can be connected to ∞ by a curve γ such that $f^n|_\gamma \to \infty$ uniformly.

But they also constructed a counterexample to the strong Eremenko's conjecture:

Theorem (Rottenfußer, Ruckert, Rempe and Schleicher, 2011)

There exists a hyperbolic entire function $f \in \mathcal{B}$ such that every path-connected component of $J(f)$ is bounded.

Note that this is not a counterexample to Eremenko's conjecture, which is still open.

[R^3S11] G. Rottenfußer, J. Ruckert, L. Rempe, and D. Schleicher, *Dynamic rays of bounded-type entire functions*, Ann. of Math. (2) 173 (2011), no. 1, 77–125.

The fast escaping set of transcendental entire functions

Let f be an entire function and, for $r > 0$, define

$$M(r) := \max_{|z|=r} |f(z)|.$$

Let $R > 0$ be sufficiently large so that $M^n(R) \to +\infty$ as $n \to \infty$. We say that $z \in A(f)$ if there exists $\ell \in \mathbb{N}$ such that

$$|f^{n+\ell}(z)| \geqslant M^n(R), \qquad \text{for all } n \in \mathbb{N}.$$

The set $A(f)$ is called the **fast escaping set of f** and is non-empty.

Theorem (Rippon and Stallard, 2005)

Let f be a transcendental entire function. The components of $A(f)$ are all unbounded.

Corollary (Rippon and Stallard, 2005)

The escaping set $I(f)$ has at least one unbounded component.

[BH99] W. Bergweiler and A. Hinkkanen, *On semiconjugation of entire functions*, Math. Proc. Cambridge Philos. Soc. 126 (1999), no. 3, 565–574.

[RS12] P. Rippon and G. Stallard, *Fast escaping points of entire functions*, Proc. London Math. Soc. (3) 105 (2012) 787–820.

Spider's web

One situation in which Eremenko's conjecture holds, is when $I(f)$ is connected. To study this, Rippon and Stallard introduced the notion of a spider's web:

We say that a set $E \subseteq \mathbb{C}$ is a **spider's web** if E is connected, and there exists a sequence $\{G_n\}_{n\in\mathbb{N}}$ of bounded, simply connected domains such that

- $\partial G_n \subseteq E$ for all $n \in \mathbb{N}$;
- $G_n \subseteq G_{n+1}$ for all $n \in \mathbb{N}$;
- $\bigcup_{n\in\mathbb{N}} G_n = \mathbb{C}$.

Theorem (Rippon and Stallard, 2005)

If f is a trancendental entire function with a multiply connected Fatou component, then both $A(f)$ and $I(f)$ are a spider's web.

Although it was proved before, this follows directly from the next result using the level sets $A_R(f)$ of the fast escaping set $A(f)$.

[RS05] P. Rippon and G. Stallard, *On questions of Fatou and Eremenko*, Proc. Amer. Math.Soc. 133 (2005) 1119–1126.

[RS12] P. Rippon and G. Stallard, *Fast escaping points of entire functions*, Proc. London Math. Soc. (3) 105 (2012) 787–820.

Level sets $A_R(f)$

Let f be a transcendental entire function and choose $R > 0$ be sufficiently large that $M^n(R) \to +\infty$ as $n \to \infty$. We define a **level set** of $A(f)$ by

$$A_R(f) := \{z \in \mathbb{C} \ : \ |f^n(z)| \geqslant M^n(R) \text{ for all } n \geqslant 0\},$$

and then

$$A(f) = \bigcup_{\ell \geqslant 0} f^{-\ell}(A_R(f)).$$

Theorem (Rippon and Stallard, 2009)

Let f be a transcendental entire function and let $R > 0$ be such that $M(r, f) > r$ for $r \geqslant R$. If $A_R(f)^c$ has a bounded component, then each of $A_R(f)$, $A(f)$ and $I(f)$ is a spider's web.

There are no examples for which $A(f)$ is a spider's web but $A_R(f)$ is not a spider's web.

[RS09] P. Rippon and G. Stallard, ***Escaping points of entire functions of small growth***, Math. Z. 261 (2009) 557–570.

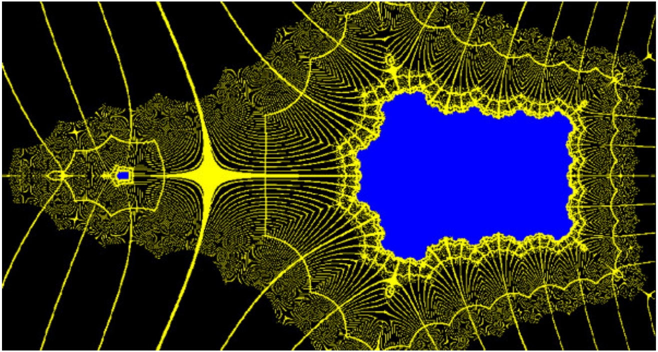
[RS12] P. Rippon and G. Stallard, ***Fast escaping points of entire functions***, Proc. London Math. Soc. (3) 105 (2012) 787–820.

First example

For the transcendental entire function

$$f(z) = \frac{1}{2}\left(\cos z^{1/4} + \cosh z^{1/4}\right),$$

the level set $A_R(f)$ (and hence $A(f)$ and $I(f)$) is a spider's web.



(picture by Dr. Dominique Fleischmann)

[RS09] P. Rippon and G. Stallard, ***Escaping points of entire functions of small growth***, **Math. Z. 261 (2009) 557–570.**

Classes of functions for which $A_R(f)$ is a spider's web

Theorem (Rippon and Stallard, 2012)

Let f be a transcendental entire function and let $R > 0$ be such that $M(r, f) > r$ for $r \geqslant R$. Then $A_R(f)$ is a spider's web if one of the following holds:

1. *f has a multiply connected Fatou component;*
2. *f has very small growth;*
3. *f has order less than $1/2$ and regular growth;*
4. *f has finite order, Fabry gaps and regular growth;*
5. *f has a sufficiently stong version of the pits effect and has regular growth.*

- Mihaljević-Brand and Peter proved that the Poincaré function (or linearizer) of some polynomials at a repelling fixed point has an $A_R(f)$ spider's web;
- Sixsmith gave other ways to construct functions with an $A_R(f)$ spider's web, some of which are preserved by composition, differentiation and integration.

[MP12] H. Mihaljević-Brandt and J. Peter, *Poincaré functions with spiders' webs*, Proc. Amer. Math. Soc. 140 (2012), no. 9, 3193–3205.

[RS12] P. Rippon and G. Stallard, *Fast escaping points of entire functions*, Proc. London Math. Soc. (3) 105 (2012) 787–820.

[Six11] D. J. Sixsmith, *Entire functions for which the escaping set is a spider's web*, Math. Proc. Cambridge Philos. Soc. 151 (2011), no. 3, 551–571.

Properties of functions with an $A_R(f)$ spider's web

Theorem (Rippon and Stallard, 2012)

Let f be a transcendental entire function and let $R > 0$ be such that $M(r, f) > r$ for $r \geqslant R$. Suppose that $A_R(f)$ is a spider's web, then:

1. *if f has no multiply connected Fatou component, then $A_R(f) \cap J(f)$, $A(f) \cap J(f)$, $I(f) \cap J(f)$ and $J(f)$ are spiders' webs;*
2. *the function f has no unbounded Fatou components;*
3. *all the components of $A(f)^c$ are compact, and if K is such a component, then either $K \cap I(f) = \emptyset$ or all points in K escape to ∞ uniformly;*
4. *every point of $J(f)$ is the limit of a sequence of points, each of which lies in a distinc component of $A(f)^c$;*
5. *each point in $I(f)$ belongs to an unbounded continuum in $I(f)$ on which all points escape to ∞ uniformly;*
6. *there is no path to ∞ on which f is bounded and so*
   - 6.1 *f does not belong to the class $\mathcal{B}$;*
   - 6.2 *f has no exceptional points (that is, points with a finite backward orbit).*

[RS12] **P. Rippon and G. Stallard, *Fast escaping points of entire functions*, Proc. London Math. Soc. (3) 105 (2012) 787–820.**

Can $I(f)$ be a SW when $A(f)$ is not?

Yes. The first example was given by Rippon and Stallard. Let $m(r) := \min_{|z|=r} |f(z)|$.

Theorem (Rippon and Stallard, 2013)

Let f be a transcendental entire function of the form

$$f(z) = cz^{p_0} \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)^{p_n},$$

where $c \in \mathbb{R} \setminus \{0\}$, $p_n \in \mathbb{N} \cup \{0\}$, for $n \geqslant 0$, and the sequence (a_n) is positive and strictly increasing. In addition, suppose that there exist $m > 1$ and $R_0 > 0$ such that, for all $r \geqslant R_0$, there exists $\rho \in (r, r^m)$ with $m(\rho) \geqslant M(r)$. Then

(a) *there are no unbounded components of $F(f)$;*

(b) *the set $I(f)$ is a spider's web, and hence is connected;*

(c) *if, in addition, f has no multiply connected Fatou components, then $J(f)$ and $I(f) \cap J(f)$ are both spiders' webs.*

It follows from another result of Rippon and Stallard that there are functions that satisfy the assumptions of this theorem for which $A_R(f)$ and $A(f)$ are not a spider's web.

[RS13a] P. J. Rippon and G. S. Stallard, ***Baker's conjecture and Eremenko's conjecture for functons with negative zeros***, J. Anal. Math. 120 (2013) 291–309.

[RS13b] P. J. Rippon and G. M. Stallard, ***A sharp growth condition for a fast escaping spider's web***, Adv. Math, 244 (2013), 337–353. [19]

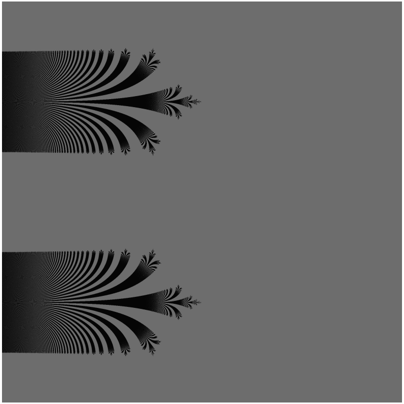
Fatou's web

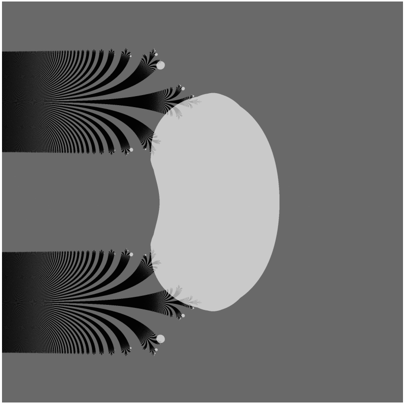
Theorem (Evdoridou, 2016)

The escaping set of Fatou's function

$$f(z) = z - 1 + e^{-z}$$

is a spider's web.





[Evd16] V. Evdoridou, ***Fatou's web***, Proc. Amer. Math. Soc. 144 (2016), no. 12, 5227–5240.

Fatou's web

For Fatou's function,

- $J(f)$ is a Cantor bouquet (an uncountable union of curves);
- $F(f)$ is a completely invariant Baker domain;
- $A(f)$ consists of the curves in the Cantor bouquet, without some of its endpoints;
- $I(f)$ consists of all $\mathbb{C}$ except for some of the endpoints of the Cantor bouquet.

Observe that neither $A_R(f)$ nor $A(f)$ can be spiders' webs.

Strategy of the proof: show that the set

$$I(f,(a_n)) := \{z \in \mathbb{C} \ : \ |f^n(z)| \geqslant a_n \text{ for all } n \in \mathbb{N}\} \subseteq I(f)$$

is a spider's web, where $a_n = (n+m)/2$ for $n \in \mathbb{N}$ ($m \in \mathbb{N}$ is arbitrary). Then, it follows from the next result from Rippon and Stallard that $I(f)$ is a spider's web.

Theorem (Rippon and Stallard, 2013)

Let f be a transcendental entire function. If $I(f)$ contains a spider's web, then $I(f)$ is a spider's web. The same holds for $J(f)$ and $I(f) \cap J(f)$.

[Evd16] V. Evdoridou, ***Fatou's web***, Proc. Amer. Math. Soc. 144 (2016), no. 12, 5227–5240.

[RS13a] P. J. Rippon and G. S. Stallard, ***Baker's conjecture and Eremenko's conjecture for functons with negative zeros***, J. Anal. Math. 120 (2013) 291–309.

Generalisation to disjoint-type maps

We say that a transcendental entire function f is of **disjoint type** if it is hyperbolic and $F(f)$ is connected.

For disjoint type functions,

- $J(f)$ is a Cantor bouquet (an uncountable union of curves);
- $F(f)$ is a completely invariant basin of attraction;
- $A(f)$ consists of the curves in the Cantor bouquet, without some of its endpoints;
- $F(f) \cup A(f)$ consists of all $\mathbb{C}$ except for some of the endpoints of the Cantor bouquet.

Theorem (Evdoridou and Sixsmith, 2019)

Let f be a disjoint-type function that is of finite order or can be written as a finite composition of finite order functions in the class $\mathcal{B}$. Then $F(f) \cup A(f)$ is a spider's web.

[ES19] V. Evdoridou and D. J. Sixsmith, *The topology of the set of non-escaping endpoints*, to appear in International Mathematics Research Notices, arXiv:1802.02738v1, 2019.

$J(f)$ spiders' webs

Theorem (Osborne, 2013)

Let f be a transcendental entire function such that $J(f)$ is locally connected. Then $J(f)$ is a spider's web.

An immediate corollary (after the previous results of Rippon and Stallard) is that:

Corollary (Osborne, 2013)

Let f be a transcendental entire function with an unbounded Fatou component. Then $J(f)$ is not locally connected.

Example (Osborne, 2013)

Let $f(z) = \sin z$. Then $J(f)$ is a spider's web, but $I(f)$ is not.

[Osb13] J. W. Osborne, *Spiders' webs and locally connected Julia sets of transcendental entire functions*, Ergod. Th. & Dynam. Sys. 33 (2013), no. 4, 1146–1161.

Transcendental self-maps of $\mathbb{C}^*$

A **transcendental self-map of** $\mathbb{C}^*$ is a holomorphic function $f : \mathbb{C}^* \to \mathbb{C}^*$ for which both 0 and ∞ are essential singularities.

We define the **escaping set** of such map as

$$I(f) := \{z \in \mathbb{C}^* \ : \ \omega(z, f) \subseteq \{0, \infty\}\},$$

where $\omega(z, f) := \bigcap_{n\in\mathbb{N}} \overline{\{f^k(z) \ : \ k \geqslant n\}}$ and the closure is taken in $\widehat{\mathbb{C}}$.

Theorem (Martí-Pete, 2018)

Let f be a transcendental self-map of $\mathbb{C}^*$. Then

(i) $I(f) \cap J(f) \neq \emptyset$;

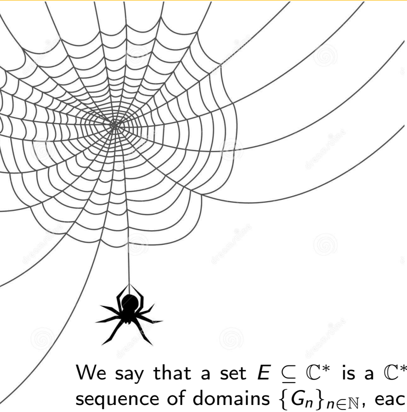
(ii) $J(f) = \partial I(f)$;

(iii) *all the components of $\overline{I(f)}$ are unbounded.*

Recall that we say that a set $X \subseteq \mathbb{C}^*$ is **bounded in** $\mathbb{C}^*$ if its closure in $\widehat{\mathbb{C}}$ does not meet $\{0, \infty\}$. Otherwise X is said to be unbounded in $\mathbb{C}^*$.

[Mar18] D. Martí-Pete, *The escaping set of transcendental self-maps of the punctured plane*, Ergod. Th. & Dynam. Sys. 38 (2018), no. 2, 739–760.

Spiders' webs in $\mathbb{C}^*$



We say that a set $E \subseteq \mathbb{C}^*$ is a $\mathbb{C}^*$-**spider's web** if E is connected and there exists a sequence of domains $\{G_n\}_{n\in\mathbb{N}}$, each of which is bounded in $\mathbb{C}^*$, such that

(a) for each $n \in \mathbb{N}$, the set $\widehat{\mathbb{C}} \setminus G_n$ has exactly two components, one containing ∞ and one containing 0;

(b) $G_n \subseteq G_{n+1}$ and $\partial G_n \subseteq E$, for $n \in \mathbb{N}$;

(c) $\bigcup_{n\in\mathbb{N}} G_n = \mathbb{C}^*$.

[EMS19] V. Evdoridou, D. Martí-Pete and D. J. Sixsmith, *Spiders' webs in the punctured plane*, preprint, arXiv:1901.05276, 2019.

Topological characterisation of a spider's web

If $E, X, Y \subseteq \widehat{\mathbb{C}}$, then we say that E **separates** Y **from** X if there exists an open set U containing Y such that the closure of U in $\widehat{\mathbb{C}}$ does not meet X, and such that $\partial U \subseteq E$.

If $Y = \{z\}$, then we say that E **separates** z **from** X.

Theorem (Evdoridou and Rempe-Gillen, 2018)

Let $E \subseteq \mathbb{C}$ be connected. Then E is a spider's web if and only if E separates every finite point $z \in \mathbb{C}$ from ∞.

Theorem (Evdoridou, Martí-Pete and Sixsmith, 2019)

Let $E \subseteq \mathbb{C}^*$ be connected. Then E is a $\mathbb{C}^*$-spider's web if and only if it separates each point of $\mathbb{C}^*$ from $\{0, \infty\}$.

[EMS19] V. Evdoridou, D. Martí-Pete and D. J. Sixsmith, *Spiders' webs in the punctured plane*, preprint, arXiv:1901.05276, 2019.

[ER18] V. Evdoridou and L. Rempe-Gillen, *Non-escaping endpoints do not explode*, Bull. Lond. Math. Soc. 50 (2018), no. 5, 916–932.

Dynamical characterisation of a spider's web

Theorem (Sixsmith, 2018)

Suppose that f is a transcendental entire function. Then $I(f)$ (respectively $A(f)$) is a spider's web if and only if it separates some point of $J(f)$ from infinity. If f has no multiply connected Fatou components, then $J(f)$ is a spider's web if and only if it separates some point of $J(f)$ from infinity.

Theorem (Evdoridou, Martí-Pete and Sixsmith, 2019)

Suppose that f is a transcendental self-map of $\mathbb{C}^*$. Then $I(f)$ is a $\mathbb{C}^*$-spider's web if and only if it separates some point of $J(f)$ from $\{0, \infty\}$. This statement is also true if we replace $I(f)$ with either $A(f)$ or $J(f)$.

Note that to prove this result, we show that:

- as for transcendental entire functions, if $I(f)$ contains a $\mathbb{C}^*$-spider's web, then $I(f)$ is a $\mathbb{C}^*$-spider's web;
- suppose that $X \subseteq \mathbb{C}^*$ is forward invariant, then X contains a $\mathbb{C}^*$-spider's web if and only if X separates some point of $J(f)$ from $\{0, \infty\}$.

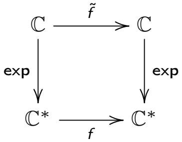
This last result is based on the blowing-up property.

[EMS19] V. Evdoridou, D. Martí-Pete and D. J. Sixsmith, *Spiders' webs in the punctured plane*, preprint, arXiv:1901.05276, 2019.

[Six18] D. J. Sixsmith, *Dynamical sets whose union with infinity is connected*, to appear in Ergod. Th. & Dynam. Sys. (published online on 2018).

Lifts

Given a holomorphic self-map of $\mathbb{C}^*$, there exists an entire function $\tilde{f}$ such that the following diagram commutes:



We call $\tilde{f}$ a lift of f, and is unique up to addition of integer multiples of $2\pi i$.

Theorem (Bergweiler, 1995)

Let f be a holomorphic self-map of $\mathbb{C}^*$ and let $\tilde{f}$ be a lift of f. Then we have $J(\tilde{f}) = \exp^{-1}(J(f))$.

Observe that $\exp^{-1} I(f) \subseteq I(\tilde{f})$, but we do not have equality.

Theorem (Evdoridou, Martí-Pete and Sixsmith, 2019)

If E is a spider's web, then $E' := \exp(E)$ is a $\mathbb{C}^*$-spider's web. Similarly, if E' is a $\mathbb{C}^*$-spider's web, then $E := \exp^{-1}(E')$ is a spider's web.

[Ber95] W. Bergweiler, *On the Julia set of analytic self-maps of the punctured plane*, Analysis 15 (1995), no. 3, 251–256.

[EMS19] V. Evdoridou, D. Martí-Pete and D. J. Sixsmith, *Spiders' webs in the punctured plane*, preprint, arXiv:1901.05276, 2019.

$J(f)$ spiders' webs in $\mathbb{C}^*$

The transcendental entire functtion $\tilde{f}(z) = \sin z$ is a lift of the transcendental self-map of $\mathbb{C}^*$ given by

$$f(z) = \exp\left(\frac{1}{2}\left(z - \frac{1}{z}\right)\right)$$

and therefore:

- since Osborne proved that $J(\tilde{f})$ is a spider's web,
- by Bergweiler's result, $J(f) = \exp(J(\tilde{f}))$,
- and our previous theorem implies that $J(f)$ is a spider's web.

Note that $\mathbb{C}^* \setminus J(f) \neq \emptyset$.

[Ber95] W. Bergweiler, ***On the Julia set of analytic self-maps of the punctured plane***, **Analysis** 15 **(1995), no. 3, 251–256.**

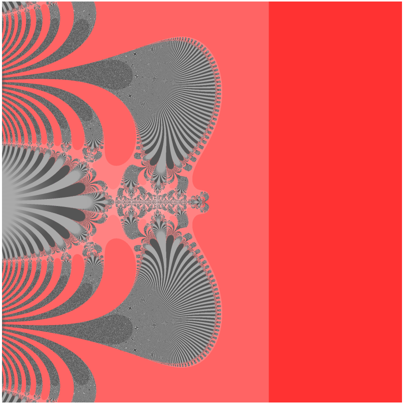
[EMS19] **V. Evdoridou, D. Martí-Pete and D. J. Sixsmith,** ***Spiders' webs in the punctured plane***, **preprint, arXiv:1901.05276, 2019.**

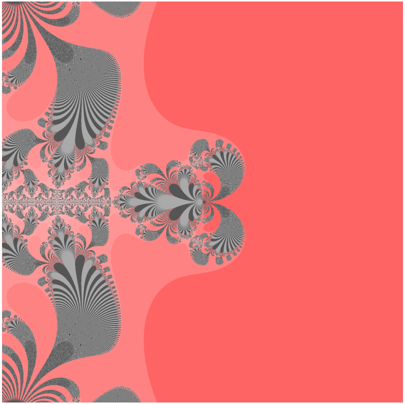
[Osb13] **J. W. Osborne,** ***Spiders' webs and locally connected Julia sets of transcendental entire functions***, **Ergod. Th. & Dynam. Sys.** 33 **(2013), no. 4, 1146–1161.**

Example of a Baker domain in $\mathbb{C}^*$

Theorem (Martí-Pete, 2019)

For $\lambda \geqslant 2$, the function $f(z) := \lambda z \exp(e^{-z}/z)$ is a transcendental self-map of $\mathbb{C}^*$ with a Baker domain containing the right half-plane $H := \{z \in \mathbb{C} \ : \ \operatorname{Re} z \geqslant 2\}$.





[Mar19] D. Martí-Pete, ***Escaping Fatou components of transcendental self-maps of the punctured plane***, to appear in Math. Proc. Camb. Phil. Soc., arXiv:1801.00124, 2019.

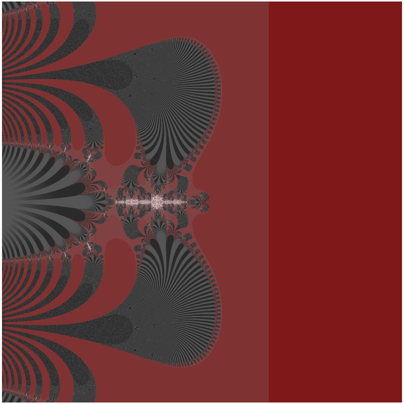
Main theorem

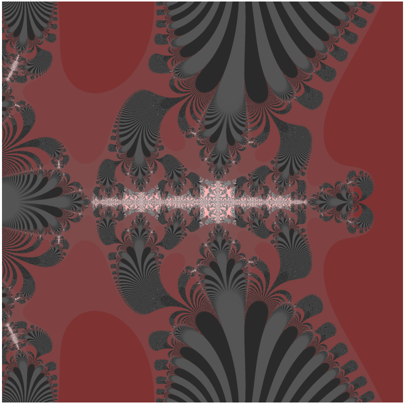
Theorem

There exists $\lambda_0 \geqslant 2$ *such that if* $\lambda \geqslant \lambda_0$*, the transcendental self-map of* $\mathbb{C}^*$ *given by*

$$f(z) := \lambda z \exp(e^{-z}/z)$$

has the property thtat $I(f)$ *is a* $\mathbb{C}^*$*-spider's web.*





[EMS19] **V. Evdoridou, D. Martí-Pete and D. J. Sixsmith, *Spiders' webs in the punctured plane*, preprint, arXiv:1901.05276, 2019.**

Sketch of the proof I

Strategy of the proof: we show that all the complementary components of the closed set

$$I := \{z \in \mathbb{C}^* \ : \ \text{for all } n \in \mathbb{N}, \ \text{either } |f^n(z)| \geqslant n/2, \ \text{or } |f^n(z)| \leqslant n/2, \ \text{or } f^{n+2}(z) \in H\},$$

which is a subset of $I(f)$, are bounded in $\mathbb{C}^*$. Here H is the absorbing half-plane

$$H := \{z \in \mathbb{C} \ : \ \operatorname{Re} z \geqslant 2\}$$

in the Baker domain of f.

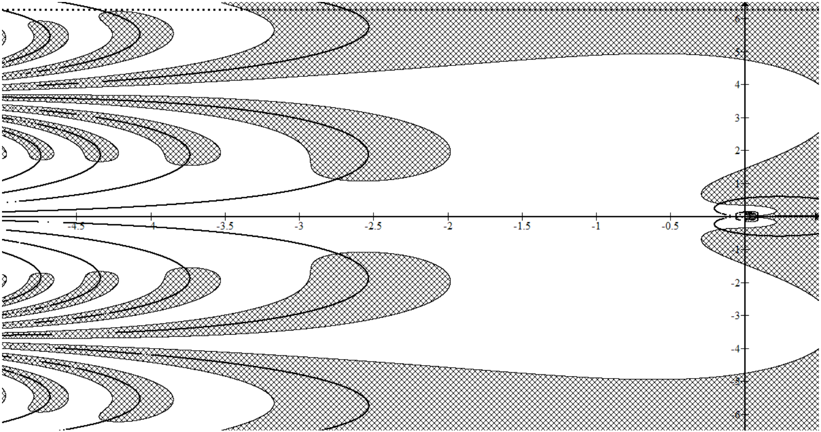
Observe that this implies our result: Pick a repelling periodic point z of f and let W be the complementary component of the closed set I containing z. Then W is a bounded domain that intersects $J(f)$ such that $\partial W \subseteq I \subseteq I(f)$. Thus, we can apply our dynamical characterisation of $\mathbb{C}^*$-spiders' webs and we are done.

Sketch of the proof II

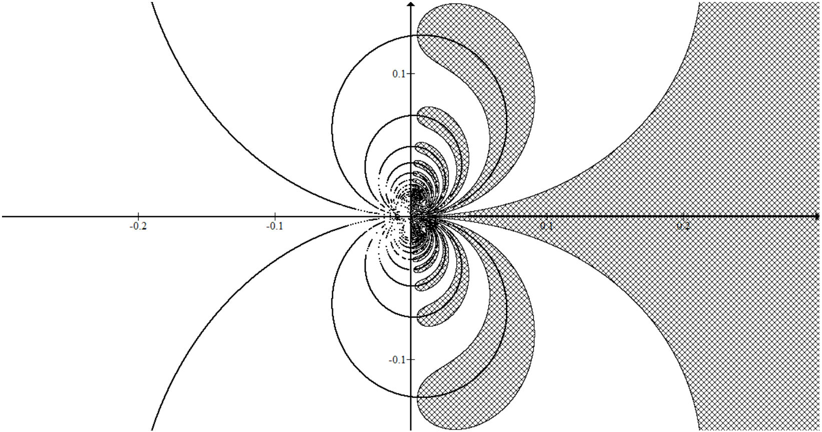
The proof that all the components of $\mathbb{C}^* \setminus I$ are bounded in $\mathbb{C}^*$ is by contradiction:

- Suppose that X is a component of $\mathbb{C}^* \setminus I$ that is unbounded in $\mathbb{C}^*$.
- Since X is open, let Γ_1 be a *long* curve contained in X.
- Note that if $z \in \mathbb{C}^* \setminus I$ and $k \in \mathbb{N}$ are such that $|f^n(z)| \geqslant n/2$ or $|f^n(z)| \leqslant 2/n$ for all $0 < n < k$, then $f^k(z) \notin f^{-2}(H)$.
- By studying the set $f^{-2}(H)$, we deduce that Γ_1 contains a possibly slightly shorter curve Γ_1' such that it is entirely contained in one of the sets where $|f|$ is very large or very small.
- We then conclude that $\Gamma_2 = f(\Gamma_1')$ is again a long curve that does not meet $f^{-2}(H)$.
- Iterating this process, we obtain a point in Γ_1 that lies in I, which is a contradiction.

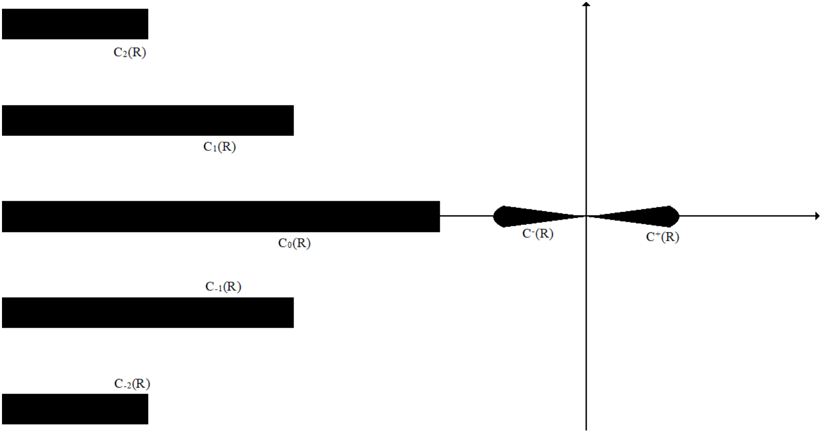
The set $f^{-2}(H)$



The set $f^{-2}(H)$ (near the origin)



The paces where $|f|$ is very large or very small



What about $A_R(f)$ or $A(f)$ spider's webs in $\mathbb{C}^*$?

One can also define an analogue of the sets $A_R(f)$ and $A(f)$ in $\mathbb{C}^*$ by combining the iterates of the minimum and maximum modulus functions (see [Mar18, Definition 1.2]).

For our function, $I(f)$ is a spider's web but $A(f)$ is not (because of the Baker domain).

It is easy to see that $A_R(f)$ can never be a spider's web in $\mathbb{C}^*$. Indeed, on every loop that is close to 0 or to ∞ the function needs to be at the same time very small and very big (as both 0 and ∞ are asymptotic values). So we necessarily have points whose next image is not too big nor too small, and so are not in $A_R(f)$.
Since all examples of transcendental entire functions for which $A(f)$ is a spider's web have been constructed using that the set $A_R(f)$ is a spider's web, we pose the following conjecture.

Conjecture (Evdoridou, Martí-Pete and Sixsmith)

Lett f be a transcendental self-map of $\mathbb{C}^*$. Then $A(f)$ is not a spider's web.

[EMS19] **V. Evdoridou, D. Martí-Pete and D. J. Sixsmith, *Spiders' webs in the punctured plane*, preprint, arXiv:1901.05276, 2019.**

Wszystkiego Najlepszego
Prof. Michał Misiurewicz!