On Möbius disjointness conjecture

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The Möbius function and its square

- $n = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$,
- $\boldsymbol{\mu}(n) = (-1)^k$ if $\alpha_1 = \ldots = \alpha_k = 1$ and $\boldsymbol{\mu}(n) = 0$ otherwise ($\boldsymbol{\mu}(1) = 1$),
- $\boldsymbol{\mu}(mn) = \boldsymbol{\mu}(m)\boldsymbol{\mu}(n)$ whenever $(m, n) = 1$ ($\boldsymbol{\mu}$ is multiplicative but not completely multiplicative),
- $\boldsymbol{\mu}^2 = \mathbb{1}_{\mathscr{S}}$, where $\mathscr{S} := \{n \in \mathbb{N} : \text{ no square divides } n\}$.

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PNT in terms of μ

Theorem (Landau, 1906)

$M(\mu) := \lim_{N\to\infty} \frac{1}{N} \sum_{n\leqslant N} \mu(n) = 0$ if and only if PNT holds.

Recall: PNT: $|\{p \leqslant N : p \text{ is prime}\}| \sim \frac{N}{\log N}$.

Recall Riemann Hypothesis is equivalent to $\sum_{n\leqslant N} \mu(n) = O(N^{\frac{1}{2}+\varepsilon})$ (for all $\varepsilon > 0$).

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Chowla conjecture

Chowla conjecture on correlations of the Möbius function (Chowla, 1965)

$$\frac{1}{N}\sum_{n\leqslant N}\boldsymbol{\mu}^{s_0}(n)\cdot\boldsymbol{\mu}^{s_1}(n+a_1)\cdot\ldots\cdot\boldsymbol{\mu}^{s_r}(n+a_r)\to 0$$

for all $r\geqslant 0$, $1\leqslant a_1<a_2<\ldots<a_r$ and $s_j\in\{1,2\}$, not all $s_0,s_1,\ldots,s_r$ equal 2.

Chowla conjecture and dynamics

- $(X_{\boldsymbol{\mu}}, S)$, the Möbius subshift ($X_{\boldsymbol{\mu}} \subset \{-1,0,1\}^{\mathbb{Z}}$), $(X_{\boldsymbol{\mu}^2}, S)$ the square-free subshift ($(X_{\boldsymbol{\mu}^2}, S) \subset \{0,1\}^{\mathbb{Z}}$).
- (X_{μ^2}, S) is a topological factor of (X_{μ}, S) (by squaring coordinatewise).
- $\frac{1}{N}\sum_{n\leqslant N} \boldsymbol{\mu}^2(n)\cdot\boldsymbol{\mu}^2(n+a_1)\cdot\ldots\cdot\boldsymbol{\mu}^2(n+a_r) \to \alpha(a_1,\ldots,a_r)$ (Mirsky, 1949).
- $\boldsymbol{\mu}^2$ is generic for the Mirsky measure $\nu_{\boldsymbol{\mu}^2}$.
- For $\theta(x) = x_0$ on $X_{\boldsymbol{\mu}} \subset \{-1,0,1\}^{\mathbb{Z}}$ we have

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- $\theta\cdot\theta\circ S^{a_1}\cdot\ldots\cdot\theta\circ S^{a_r} \in C(X_{\boldsymbol{\mu}})$, $\boldsymbol{\mu} \in X_{\boldsymbol{\mu}}$.
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Definition (Möbius disjointness)

Let X be a compact metric space and $T\colon X \to X$ a homeomorphism of it. One says that (X, T) is *Möbius disjoint* if $\lim_{N\to\infty} \frac{1}{N} \sum_{n\leqslant N} f(T^n x)\mu(n) = 0$ for all $x \in X$, $f \in C(X)$.

Sarnak's conjecture (2010)

All zero topological entropy dynamical systems (X, T) are Möbius disjoint.

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What makes it a difficult problem?

Assume that $(X, \mathcal{B}, \mu, T)$ is a measure-theoretic dynamical system.

- $\left\| \frac{1}{N} \sum_{n \leqslant N} f(T^n x)\boldsymbol{\mu}(n) \right\|_{L^2(X,\mu)} = \left\| \frac{1}{N} \sum_{n \leqslant N} z^n \boldsymbol{\mu}(n) \right\|_{L^2(\mathbb{S}^1, \sigma_f)}$;
- $\sup_{z \in \mathbb{S}^1} \left| \sum_{n \leqslant N} z^n \boldsymbol{\mu}(n) \right| \leqslant C_A \frac{N}{\log^A N}$ for some $C_A > 0$ and all $N \geqslant 2$ (for each $A > 0$, Davenport 1937);
- L^2-version of Möbius disjointness holds always;
- Using Davenport's estimate: *Given $f \in L^1(X, \mu)$, for a.e. $x \in X$, we have* $\frac{1}{N} \sum_{n \leqslant N} f(T^n x)\boldsymbol{\mu}(n) \to 0$ (Sarnak 2010).

Answer to the question

$\frac{1}{N} \sum_{n \leqslant N} f(T^n x)\boldsymbol{\mu}(n) \to 0$ for all $f \in C(X)$ and all $x \in X$.

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Relation between conjectures of Chowla and Sarnak

Conjectures of Chowla and Sarnak (Sarnak 2010; Tao 2012)

Chowla conjecture $\Rightarrow$ Sarnak's conjecture.

Recall of Sarnak's conjecture: $\frac{1}{N}\sum_{n\leqslant N} f(T^n x)\boldsymbol{\mu}(n) \to 0$ for each deterministic homeomorphism T (of a compact metric space X) and all $f \in C(X)$ and $x \in X$.

But why ergodic theory?

- $\frac{1}{N}\sum_{n\leqslant N} f(T^n x)\boldsymbol{\mu}(n) = \frac{1}{N}\sum_{n\leqslant N} f(T^n x)\theta(S^n\boldsymbol{\mu}) = \int_{X\times X_{\boldsymbol{\mu}}}(f\otimes\theta)d\left(\frac{1}{N}\sum_{n\leqslant N}\delta_{(T\times S)^n(x,\boldsymbol{\mu})}\right)$ (Notation: $\theta(z)=z(0)$ for $z\in X_{\boldsymbol{\mu}}\subset\{-1,0,1\}^{\mathbb{Z}}$),
- $\frac{1}{N_k}\sum_{n\leqslant N_k}\delta_{(T\times S)^n(x,\boldsymbol{\mu})}\to\rho$ in the space $M(X\times X_{\boldsymbol{\mu}})$,
- ρ is $T\times S$-invariant; projection of ρ on $X_{\boldsymbol{\mu}}$ is equal to $\widehat{\nu}_{\boldsymbol{\mu}^2}$ (since $\boldsymbol{\mu}$ is a generic point for $\widehat{\nu}_{\boldsymbol{\mu}^2}$ UNDER the Chowla conjecture(!)); projection of ρ on X is SOME T-invariant measure κ and $h_\kappa(T)=0$ by the variational principle.

“Joining” prof of the implication “Chowla $\Rightarrow$ Sarnak” (Abdalaoui, Kułaga-Przymus, L., de la Rue, 2013)

ρ is a *joining* of the dynamical system (X,κ,S) and $(\{-1,0,1\}^{\mathbb{Z}},\widehat{\nu}_{\boldsymbol{\mu}^2},S)$. The latter automorphism has the so called relative Kolmogorov property with respect to the factor $(X_{\boldsymbol{\mu}^2},\nu_{\boldsymbol{\mu}^2},S)$ given by the square free numbers and the Mirsky measure $\nu_{\boldsymbol{\mu}^2}$. Then one uses some elements of disjointness theory of Furstenberg (relative version).

But, does Sarnak's conjecture imply the Chowla conjecture?

- Logarithmic Chowla conjecture:

$$\frac{1}{\log N}\sum_{n\leqslant N}\frac{\mu^{s_0}(n)\mu^{s_1}(n+a_1)\ldots\mu^{s_r}(n+a_r)}{n}\to 0$$

for all $r\geqslant 0$, $1\leqslant a_1<\ldots<a_r$ and $s_j\in\{1,2\}$, not all $s_0,\ldots,s_r$ równe 2.

- Logarithmic Sarnak's conjecture: $\frac{1}{\log N}\sum_{n\leqslant N}\frac{f(T^nx)\mu(n)}{n}\to 0$ for each deterministic (X,T) and all $f\in C(X)$ and $x\in X$.
- Chowla implies logarithmic Chowla conjecture; Sarnak's conjecture implies logarithmic Sarnak's conjecture.

Theorem (Tao, 2016)

The logarithmic Chowla conjecture holds for correlations of length 2.

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The logarithmic Chowla conjecture holds for all corellations of ODD length.

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Logarithmic Chowla conjecture is EQUIVALENT to logarithmic Sarnak's conjecture.

Corollary (Gomilko, Kwietniak, L., 2017)

Sarnak's conjecture implies Chowla conjecture along a subsequence.

- (Tao, 2017) If Sarnak's conjecture holds then Chowla conjecture holds along a subsequence of full logarithmic density.
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How do people show Möbius disjointness in selected classes?

- $\boldsymbol{\mu}$ is aperiodic (classical), that is: $\frac{1}{N}\sum_{n\leqslant N}\boldsymbol{\mu}(an+b)\to 0$ for each $a,b\geqslant 1$. Check that each periodic system is Möbius disjoint; this can be used in a weaker version when a system is "approximated" by periodic systems: Karagulyan for zero entropy interval maps, 2013; subshifts given by regular Toeplitz sequences: El Abdalaoui, Downarowicz, Kasjan, L., 2013.
- Daboussi-Kátai-Bourgain-Sarnak-Ziegler criterion (DKBSZ): $(f_n)\subset\mathbb{C}$ bounded such that $\frac{1}{N}\sum_{n\leqslant N} f_{pn}\overline{f}_{qn}\to 0$ for each different primes p,q sufficiently large then $\sum_{n\leqslant N} f_n\boldsymbol{u}(n)\to 0$ for EACH bounded multiplicative $\boldsymbol{u}:\mathbb{N}\to\mathbb{C}$.

  In the dynamical context (X,T) we use it for $f_n=f(T^nx)$ (Bourgain, Sarnak, Ziegler - Möbius disjointness of horocycle flows, 2013). Take $Tx=x+\alpha$ an irrational rotation, $f(x)=e^{2\pi ikx}$ with $k\neq 0$, and note that:

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  In the dynamical context (X,T) we use it for $f_n=f(T^nx)$ (Bourgain, Sarnak, Ziegler - Möbius disjointness of horocycle flows, 2013). Take $Tx=x+\alpha$ an irrational rotation, $f(x)=e^{2\pi ikx}$ with $k\neq 0$, and note that:

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How do people show Möbius disjointness in selected classes?

- Weakness: problem of Möbius disjointness in ALL uniquely ergodic models of an ergodic system.
  - All u.e. models of the one-point system are Möbius disjoint.
  - What about all u.e. models of the periodic system on two points?
  - What about all u.e. models of an irrational rotation? For example when it is topologically mixing (so no chance to have an eigenfunction continuous)...
- Strength: DKBSZ is weaker than Furstenberg disjointness of powers T^p and T^q - classical theory of joinings,
- Matomäki and Radziwiłł, 2015: For the Möbius function we have cancelations on typical short intervals: $\frac{1}{M}\sum_{m\leqslant M}\left|\frac{1}{H}\sum_{h\leqslant H}\mu(m+h)\right| \to 0$ when $H, M \to \infty$ and $H = \mathrm{o}(M)$.

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Möbius disjointness - general results

Theorem (El Abdalaoui, L., de la Rue, 2015)

Möbius disjointness holds for ALL uniquely ergodic models of totally ergodic rotations.

- Via introduction of a joining counterpart of DKBSZ.
- Applies to quasi-discrete spectrum, nil-rotations and other (Flaminio, Frączek, Kułaga-Przymus, L., 2017).

Theorem (Huang, Wang, Zhang, 2016)

Möbius disjointness holds for each (X, T) for which EACH invariant measure yields a measure-theoretic system with discrete spectrum.

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- In fact, as shown later by Ferenczi, Kułaga-Przymus and L., an interpretation of a result by Matomäki, Radziwiłł and Tao from 2015 gives that the spectral measure of the function θ in each Furstenberg system must be continuous. Discrete spectrum result immediately follows.

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Theorem (Huang, Wang, Ye, 2017)

Möbius disjointness holds for all systems with sub-polynomial complexity (i.e. smaller than n^δ, for each $\delta > 0$) for each invariant measure μ.

- The measure complexity of μ is weaker than n^δ if

$$\liminf_{n\to\infty} \frac{\min\{m \geqslant 1 : \mu(\bigcup_{j=1}^{n} B_{d_n}(x_i,\varepsilon)) > 1-\varepsilon \text{ for some } x_1,\ldots,x_m \in X\}}{n^\delta} = 0$$

for each $\varepsilon > 0$ (here $d_n(y,z) = \frac{1}{n}\sum_{j=1}^{n} d(T^j y, T^j z)$).

- All examples either with all invariant measures giving rise to discrete spectrum or C^∞-Anzai skew products.

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Logarithmic Sarnak's conjecture - a break-through

Theorem (Frantzikinakis, Host, 2017)

Assume that (X, T) is a zero entropy dynamical system whose set $M^e(X, T)$ of ergodic invariant measures is countable. Then (X, T) is **logarithmically** Möbius disjoint.

- Tao's identities (on infinite averages of sequences) used in his proof of logarithmic Chowla for correlations of length 2.
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From logarithmic Sarnak's conjecture to Sarnak's conjecture

Theorem (Gomilko, L., de la Rue, 2019)

If (X, T) satisfies the logarithmic strong MOMO property, then there exists $A = A(X, T) \subset \mathbb{N}$ of **full logarithmic density** such that for each $f \in C(X)$ and $x \in X$, we have

$$\frac{1}{N}\sum_{n\leqslant N} f(T^n x)\mu(n) \to 0 \text{ along } A \ni N \to \infty.$$

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- Logarithmic Sarnak holds if and only if logarithmic strong MOMO holds for all zero entropy systems.
- All systems of zero entropy having only countably many ergodic invariant measures satisfy the logarithmic strong MOMO property.

Corollary

If (X, T) has zero entropy and $M^e(X, T)$ is countable then the system is Möbius disjoint in full logarithmic density.

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Rigidity and Möbius disjointness

Theorem (Kanigowski, L., Radziwiłł, 2019)

If each measure $\mu \in M(X, T)$ yields a rigid measure-theoretic system (i.e. there exists (q_n) such that $\mu(T^{q_n}A \triangle A) \to 0$ for each Borel $A \subset X$) then (X, T) is Möbius disjoint.

- Strengthening of the main result in Matomäki- Radziwiłł from 2015, to short interval behaviour along arithmetic progressions.
- Yields Möbius disjointness of **all** 3-IETs (previously known only in the a.e. versions, Bourgain 2011, Chaika-Eskin 2017). Yields Möbius disjointness of a.e. IET.

Theorem (Kanigowski, L., Radziwiłł, 2019)

If each **ergodic** invariant measure of (X, T) yields a rigid system and there are only countably many ergodic measures then (X, T) is Möbius disjoint.