Rigidity of negatively curved manifolds

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KWIAT

(M, g) compact, boundaryless, C^∞ Riemannian manifold with negative sectional curvature and dimension d.

(M, g) is *locally symmetric* if, locally, the geodesic symmetries are isometries.

Equivalently, the group G of isometries of the universal cover $(\widetilde{M}, \widetilde{g})$ is a semi-simple Lie group with real rank 1, the Killing form on $\mathfrak{G}$ defines the Riemannian $\widetilde{g}$ metric on G/K, K maximal compact subgroup, and $(M, g) = \Gamma \backslash (G/K, \widetilde{g})$ for a cocompact torsion free discrete group Γ.

Our Game *Recognising locally symmetric spaces through global properties.*

Ex. 1 *Asymptotically harmonic* manifolds.

Let $(\widetilde{M}, \tilde{g})$ be the universal cover of (M, g).
$(\widetilde{M}, \tilde{g})$ is a Hadamard manifold, two geodesic rays in $\widetilde{M}$ are said to be equivalent if $\sup_{t \geqslant 0} d(\gamma_1(t), \gamma_2(t)) < \infty$.
The space of equivalence classes is the *boundary at infinity* $\partial\widetilde{M}$.
$S\widetilde{M}$ is identified with $\widetilde{M} \times \partial\widetilde{M}$.

Fix $\xi \in \partial\widetilde{M}$. $\widetilde{M} \times \{\xi\}$ is the *stable manifold* $\widetilde{W}^s(v)$

$$\widetilde{W}^s(v) \ := \ \{w : \sup_{t \geqslant 0} d(\varphi_t w, \varphi_t v) < +\infty\}.$$

$\widetilde{W}^s(v)$ is endowed with the metric $\tilde{g}$.

Busemann function: Fix $(x, \xi) \in S\widetilde{M}, y \in \widetilde{M}$,

$$b_{x,\xi}(y) \ := \ \lim_{z \to \xi} \left(d(y, z) - d(x, z) \right).$$

The level set $\{(y, \xi) : b_{x,\xi}(y) = 0\}$ is the *strong stable manifold*

$$\widetilde{W}^{ss}(x, \xi) := \{(y, \xi) : \lim_{t \to \infty} d(\gamma_{x,\xi}(t), \gamma_{y,\xi}(t)) = 0\}.$$

$\widetilde{W}^{ss}(x, \xi)$ projects onto

$$W^{ss}(v) = \{w \in SM : \lim_{t \to \infty} d(\gamma_w(t), \gamma_v(t)) = 0\}.$$

(M, g) is called *asymptotically harmonic* if the function $B(x, \xi)$ is constant,

$$B(x, \xi) := \Delta_y b_{x,\xi}(y)|_{y=x} = -\mathrm{Div}^s \overline{X}(v).$$

Theorem (M, g) *is asymptotically harmonic (i.e. the function* $\Delta_y b_{x,\xi}(y)|_{y=x}$ *is independent of* (x, ξ)*) if, and only if, the space* (M, g) *is locally symmetric.*

The proof combines the works of Benoist-Foulon-Labourie (92), Foulon-Labourie (92) and Besson-Courtois-Gallot (95).

Example $\widetilde{M} = \mathbb{H}^2_{\mathbb{R}}$, $b_{o,\xi}(y)$ is $-\log$(Poisson kernel). Then, $\Delta_y b_{o,\xi}(y)|_{y=x} = \|\nabla_y b_{o,\xi}(y)|_{y=x}\|^2 = 1$.

Ex. 2 *Volume entropy* $V(g)$
Let $(\widetilde{M}, \tilde{g})$ be the universal cover of (M, g),

$$V(g) := \lim_{R\to\infty} \frac{1}{R} \log \mathrm{Vol} B_{\widetilde{M}}(x, R).$$

$H(g)$:= the measure entropy of the geodesic flow for the Liouville measure m_L.

Fact $H \leqslant V$, *with equality for locally symmetric spaces.*

Katok conjecture $H = V$ *only for locally symmetric spaces.*

Conjecture $H = V$ *only for locally symmetric spaces.*
True for surfaces (Katok 82).

More generally for $g = e^{2\varphi} g_0$, (M, g_0) locally symmetric.

True in a C^2 neighborhood of (M, g_0) with constant curvature (Flaminio 97).

Open in general.

Remark $H = \int B \, dm_L$, by Pesin formula.
$V = \int B \, dm_{BR}$, where m_{BR} is the *Burger-Roblin measure.*
Katok conjecture reduces to $\int B \, dm_L = \int B \, dm_{BR}$ if, and only if, B is constant.

Burger-Roblin measure m_{BR}.

Consider $S_xM \subset SM$, φ_t the geodesic flow and for $r > 0, m_r$ the normalized Lebesgue measure on $\varphi_{-r}S_xM$. Then, (Margulis, Knieper)

$$m_{BR} := \lim_{r \to \infty} m_r.$$

Fact *Fix x_0. There exists a measure ν on $\partial\widetilde{M}$ such that, locally:*

$$dm_{BR}(x,\xi) = e^{-Vb_{x_0,\xi}(y)} d\nu(\xi) d\mathrm{Vol}(y).$$

In particular, for any C^1 vector field Z on SM such that $Z(v)$ is tangent to $W^s(v)$ for all $v \in SM$,

$$\int_{SM} \mathit{Div}^s Z(v) + V < Z(v), \overline{X}(v) > \, dm_{BR}(v) \; = \; 0. \quad (1)$$

$$\int_{SM} \mathrm{Div}^s Z(v) + V < Z(v), \overline{X}(v) > \; dm_{BR}(v) \;=\; 0. \qquad (1)$$

Corollary 1 $\int B \, dm_{BR} = V$. (Apply (1) to $Z = \overline{X}$.)

Corollary 2 *The measure* m_{BR} *is stationary for the operator* $\Delta^s + V\overline{X}$*: for* $F \in C(SM)$*,*

$$\int_{SM} (\Delta^s + V\overline{X}) F \, dm_{BR} \;=\; 0.$$

(Apply (1) to $Z = \nabla^s F$.)

$$\int_{SM} \mathrm{Div}^s Z(v) + V < Z(v), \overline{X}(v) > \ dm_{BR}(v) \ = \ 0. \qquad (1)$$

Corollary 3 *The measure m_{BR} is the unique stationary probability for the Laplacian Δ^{ss} along the strong stable foliation W^{ss}.*

(Apply (1) to $Z(v) = \frac{d}{dt} F(\varphi_t v)|_{t=0} \overline{X}(v)$. Uniqueness follows from Kaimanovich and Bowen-Marcus.)

Theorem [L.-Shu, (18), (19)]
There exists a family of probability measures $m_\rho, -\infty < \rho < V$, *such that:*

1. $m_\rho \to m_{BR}$ *as* $\rho \to V$,

2. $m_\rho \to m_L$ *as* $\rho \to -\infty$,

3. $\int B\,dm_\rho \leqslant \int B\,dm_{BR} = V$, *with equality at any* $\rho, -\infty < \rho < V$ *if, and only if, the function* B *is constant.*

Proof of 1. and 3. ([L-Shu 19])
For $-\infty < \rho < V, \Delta^s + \rho\overline{X}$ admits a unique stationary measure m_ρ. (Hamenstädt 97)

Fix x_0. There exist a measure ν^ρ on $\partial\widetilde{M}$ and a function $K^\rho_{x,\xi}(y)$ such that, locally:

$$dm_\rho(x,\xi) = K^\rho_{x_0,\xi}(y)d\nu^\rho(\xi)d\mathrm{Vol}(y).$$

$y \mapsto K^\rho_{x,\xi}(y)$ is C^∞, $(x,\xi) \mapsto K^\rho_{x_0,\xi}(y), \nabla_y K^\rho_{x_0,\xi}(y)$ are Hölder continuous (Garnett, Hamenstädt).

In particular, for any C^1 vector field Z on SM such that $Z(v)$ is tangent to $W^s(v)$ for all $v \in SM$,

$$\int_{SM} \mathrm{Div}^s Z - < Z, \nabla^s \log K^\rho > \, dm_\rho = 0. \qquad (2)$$

By (2) applied to $Z = \overline{X}$, we get

$$\int B\, dm_\rho = -\int < \overline{X}, \nabla^s \log K^\rho > \, dm_\rho,$$

and, by Schwarz inequality,

$$(\int B\, dm_\rho)^2 \leqslant \int_{SM} \|\nabla^s \log K^\rho_{x,\xi}\|^2 \, dm_\rho, \qquad (3)$$

with equality only if $\nabla^s \log K^\rho = \tau(\rho)\overline{X}$ for some $\tau(\rho)$.

The terms in the inequality (3) have a geometric interpretation related to the *diffusion* associated to the Markov operator $\Delta^s + \rho\overline{X}$.

Recall that m_ρ is the unique $\mathcal{L}^\rho$-stationary mesure, where

$$\mathcal{L}^\rho_\xi F(x,\xi) = \Delta^s_y F(y,\xi)|_{y=x} + \rho < \overline{X}, \nabla^s_y F(y,\xi)|_{y=x} >_{x,\xi} .$$

It defines a diffusion process $\omega_t, t \geqslant 0$ with the property that the trajectory $\omega_t \in W^s(\omega_0)\,\forall t$.

The *linear drift* is defined by the a.e. limit

$$\ell_\rho := \lim_{t\to\infty} \frac{1}{t} b_{\widetilde{\omega}(0)}(\widetilde{\omega}(t)) \ = \ -\rho + \int B\, dm_\rho.$$

$$\ell_\rho = \lim_{t\to 0^+} \frac{1}{t} \mathbb{E}_{m_\rho} b_{\widetilde{\omega}(0)}(\widetilde{\omega}(t)) = \int (\Delta^s + \rho \overline{X})_y b_{x,\xi}(y)\Big|_{y=x} dm_\rho(x,\xi).$$

The *stochastic entropy* (Kaimanovich 86) is defined by the a.e. limit

$$h_\rho \;:=\; \lim_{t\to\infty} -\frac{1}{t}\log p^\rho(t,\widetilde{\omega}(0),\widetilde{\omega}(t)),$$

where $\widetilde{\omega}_t, t \geqslant 0$ is the lifted trajectory to $S\widetilde{M}$ and $p^\rho(t,\widetilde{v},\widetilde{w})$ the corresponding heat kernel. We have

$$h_\rho = \int_{SM}\left(\|\nabla^s \log K^\rho_{x,\xi}\|^2 - \rho B(x,\xi)\right)\, dm_\rho$$

and the *fundamental inequality* (Guivarc'h)

$$h_\rho \;\leqslant V\ell_\rho. \qquad (4)$$

$$h_\rho = \int_{SM} \Big(\|\nabla^s \log K^\rho_{x,\xi}\|^2 - \rho B(x,\xi) \Big)$$
$$\ell_\rho = -\rho + \int B\, dm_\rho \quad \text{and} \quad h_\rho \leqslant V \ell_\rho.$$

Proof of 3. By Schwarz inequality (3),

$$h_\rho \geqslant \left(\int B\, dm_\rho\right)^2 - \rho\left(\int B\, dm_\rho\right) = \ell_\rho\left(\int B\, dm_\rho\right).$$

Proof of 1. Let m be a limit of $m_{\rho_n}, \rho_n \to V$.

$$\liminf_{\rho \to V} \int B\, dm_\rho = \liminf_{\rho \to V} (\rho + \ell_\rho) \geqslant V,$$
$$\int B\, dm = V, \quad \lim_n \ell_{\rho_n} = 0, \quad \lim_n h_{\rho_n} = 0$$

$$\int B\,dm = V, \quad \lim_n \ell_{\rho_n} = 0, \quad \lim_n h_{\rho_n} = 0$$

Then, if $Z_n := \nabla^s \log K^{\rho_n}_{x,\xi} - (\int B\,dm_{\rho_n})\overline{X}$, then

$$\lim_n \int \|Z_n\|^2\, dm_{\rho_n} = 0.$$

It follows that the measure m is stationary for Δ^{ss}, and thus has to coincide with m_{BR}.

Proof of 2. Let m be a weak*-limit of m_ρ as $\rho \to -\infty$. We want to show that $m = m_L$. Set $\rho = -\frac{1}{\varepsilon^2}$; m_ρ is stationary for $-\overline{X} + \varepsilon^2 \Delta^s$.

As $\rho \to -\infty, \varepsilon \to 0$, and m is stationary for $-\overline{X}$, i.e. m is invariant under the geodesic flow.

By Bowen and Ruelle, the Liouville measure m_L is characterized as the only geodesic flow invariant measure m' that satisfies

$$h_{m'} = \int_{SM} (D_v \varphi_{-1})\big|_{E^s(v)} \, dm'(v).$$

Remains to prove that $h_m = \int_{SM} (D_v\varphi_{-1})\big|_{E^s(v)} dm(v)$. The proof in (L.-Shu 18) follows 5 steps:

1. The construction of a stochastic flow with stationary measure m_ρ (after Elworthy, we have to go to a bigger space).

2. The definition of the relative entropy for a stochasic flow (after Kifer).

3. Pesin formula for random diffeos (follows Mañé and Liu-Shu).

4. Continuity of the RHS (from SDE theory)

5. Uper-semi-continuity of the entropy (after Yomdin. The C^∞ hypothesis is essential here).

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