Entropy density of ergodic measures for B-free shifts

Michal Kupsa (joint work with Dominik Kwietniak and Jakub Konieczny)

Institute of Information Theory and Automation, Czech Academy of Science, Prague, Czech Republic

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Möbius function, motivation for/to $\mathcal{B}$-free

Möbius function $\mu(n)$:

$$\mu(n) = \begin{cases} (-1)^k, & \text{if } n = p_1 p_2 \cdot p_k \text{ for distinct primes } p_i, \\ 0, & \text{otherwise.} \end{cases}$$

Square-free function $\eta(n)$:

$$\eta(n) = \begin{cases} 1, & \text{if } n = p_1 p_2 \cdot p_k \text{ for distinct primes } p_i, \\ 0, & \text{otherwise.} \end{cases}$$

$\eta(n) = \eta_{\mathcal{B}}(n)$, where $\mathcal{B} = \{m^2 \mid m \in \mathbb{N}\}$
($\mathcal{B} = \{p^2 \mid p \text{ is a prime}\}$ suffices)

$\mathcal{B}$-free shifts and hereditary closure

$\mathcal{B} \subset \mathbb{N}$,

- $\mathcal{B} \cdot \mathbb{N}$ - all multiples,
- $\mathcal{F}(\mathcal{B}) = \mathbb{N} \setminus \mathcal{B} \cdot \mathbb{N}$ - multiple-free set, elements are "$\mathcal{B}$-free".

Definition

Let $\eta_{\mathcal{B}} = 1_{\mathcal{F}(\mathcal{B})} \in \{0,1\}^{\mathbb{N}}$. The orbit closure $X_{\mathcal{B}} \subset \{0,1\}^{\mathbb{N}}$ of $\eta_{\mathcal{B}}$ is called $\mathcal{B}$-free shift (we consider shift map on $\{0,1\}^{\mathbb{N}}$)

Definition

The hereditary closure of a set $X \subset \{0,1\}^{\mathbb{N}}$ is defined as follows:

$$\tilde{X} = \{(y_n) \in \{0,1\}^{\mathbb{N}} \mid \exists (x_n) \in X, y_n \leq x_n, n \in \mathbb{N}\}.$$

A set is hereditary if it coincides with its hereditary closure.

A little bit more hierarchy for $\mathcal{B}$-free sets

Theorem (consequence of CRT)

If $\mathcal{B} \subset \mathbb{N}$ consists of pairwise coprime numbers, $\tilde{X}_{\mathcal{B}} = X_{\mathcal{B}}$.

There are many examples when $X_{\mathcal{B}} \subsetneq \tilde{X}_{\mathcal{B}}$. We introduce even larger shifts:

$$\tilde{X}_{\mathcal{B}}^{(k)} = \bigcap_{\mathcal{B}' \subset \mathcal{B}, \# \mathcal{B}' = k} \tilde{X}_{\mathcal{B}'}, \qquad \tilde{X}_{\mathcal{B}}^{*} = \bigcap_{\mathcal{B}' \subset \mathcal{B}, \mathcal{B}' \text{ finite}} \tilde{X}_{\mathcal{B}'}.$$

All defined sets are closed, shift-invariant and hereditary. Moreover,

$$X_{\mathcal{B}} \subset \tilde{X}_{\mathcal{B}} \subset \tilde{X}_{\mathcal{B}}^{*} \subset \ldots \subset \tilde{X}_{\mathcal{B}}^{(2)} \subset \tilde{X}_{\mathcal{B}}^{(1)}.$$

The set $\tilde{X}_{\mathcal{B}}^{(1)}$ is often called the $\mathcal{B}$-admissible shift and its elements $\mathcal{B}$-admissible sequences. We can make even finer hierarchy, taking for every countable anti-chain $\mathfrak{B}$ in the powerset of $\mathcal{B}$, the intersection of $\tilde{X}_{\mathcal{B}'}$, $\mathcal{B}' \in \mathfrak{B}$.

In the literature, $X_{\mathcal{B}}$, $\tilde{X}_{\mathcal{B}}$ and $\tilde{X}^{(1)}_{\mathcal{B}}$ have been studied. By the CRT, these sets are all the same if $\mathcal{B}$ is infinite and consists of pairwise coprime numbers.
On the other hand, examples of $\mathcal{B}$ for which the sets differ were provided, see [A. Bartnicka, S. Kasjan, J. Kulaga-Przymus and M. Lemanczyk, preprint].

We add $\tilde{X}^{*}_{\mathcal{B}}$ to this picture.

Theorem

There exists $\mathcal{B}$ such that $\tilde{X}^{*}_{\mathcal{B}} \neq \tilde{X}^{(1)}_{\mathcal{B}}$. There exists $\mathcal{B}$ such that $\tilde{X}^{*}_{\mathcal{B}} \neq \tilde{X}_{\mathcal{B}}$.

For a subshift X, we denote the set of all shift-invariant probability measures by $\mathcal{M}(X)$ and the set of all ergodic measures by $\mathcal{M}^e(X)$.

If not stated otherwise, we consider weak*-topology on these sets.

Definition

We say that the ergodic measures of a subshift X are entropy-dense (in the set of all shift-invariant measures), if for every $\mu \in \mathcal{M}(X)$ there exists a sequence of ergodic measures ν_n such that

- *ν_n converges to μ in the weak*-topology and*
- *$h(\nu_n)$ converges to $h(\mu)$.*

Theorem

For every $\mathcal{B} \subset \mathbb{N}$, the ergodic measures $\mathcal{M}^e(\tilde{X}_{\mathcal{B}})$ are entropy-dense in the set $\mathcal{M}(\tilde{X}_{\mathcal{B}})$. Moreover,

$$\mathcal{M}(\tilde{X}_{\mathcal{B}}) = \mathcal{M}(\tilde{X}^*_{\mathcal{B}}), \qquad \mathcal{M}(\tilde{X}_{\mathcal{B}}) = \mathcal{M}(\tilde{X}^*_{\mathcal{B}}).$$

Known result

Theorem (J. Kulaga-Przymus, M. Lemanczyk and B. Weiss 2016)

For every $\mathcal{B} \subset \mathbb{N}$, the ergodic measures $\mathcal{M}^e(\tilde{X}_{\mathcal{B}})$ are dense in the set $\mathcal{M}(\tilde{X}_{\mathcal{B}})$.

On the other hand, density in the weak* topology does not ensure the convergence of entropies. The entropy is not continuous w.r.t. the topology.

Theorem (T. Downarowicz and S. Kasjan 2003)

For any Choquet simplex K and any u.s.c. affine $f : K \to [0, \infty)$, there exists an expansive dynamical system (X, T) (a Toeplitz flow) and an affine homeomorphism π from $\mathcal{M}(X, T)$ to K such that $h(\mu) = f(\pi(\mu))$, $\mu \in \mathcal{M}(X, T)$.

Our main tool is $\bar{d}$-metric (more generally $\bar{f}$-metric):

$$\bar{d}(x,y) = \limsup_{n\to\infty} \frac{1}{n} \sum_{0\le i<n} |x_i - y_i| \qquad x,y \in \{0,1\}^{\mathbb{N}},$$

$$\bar{d}(\mu,\nu) = \inf_{\eta\in J(\mu,\nu)} \int_{x,y} \bar{d}(x,y)d\eta(x,y) \qquad \mu,\nu \in \mathcal{M}\left(\{0,1\}^{\mathbb{N}}\right),$$

where $J(\mu,\nu)$ is the set of all joinings of μ and ν.

We extend this distance/pseudo-distance onto sets in the following way:

$$\bar{d}(X,Y) = \max(\sup_{x\in X}\inf_{y\in Y} \bar{d}(x,y), \sup_{y\in Y}\inf_{x\in X} \bar{d}(x,y)),$$

$$\bar{d}(\mathcal{M}',\mathcal{M}'') = \max(\sup_{\mu\in\mathcal{M}'}\inf_{\nu\in\mathcal{M}''} \bar{d}(\mu,\nu), \sup_{\nu\in\mathcal{M}''}\inf_{\mu\in\mathcal{M}'} \bar{d}(\mu,\nu)),$$

where X, Y are subsets of $\{0,1\}^{\mathbb{N}}$ and $\mathcal{M}'$ and $\mathcal{M}''$ are subsets of $\mathcal{M}\left(\{0,1\}^{\mathbb{N}}\right)$.

Theorem (main scheme)

Suppose that X and X_k, $k \in \mathbb{N}$ are subshifts such that $\bar{d}(\mathcal{M}^e(X), \mathcal{M}^e(X_k))$ goes to zero, and for every $k \in \mathbb{N}$, the ergodic measures $\mathcal{M}^e(X_k)$ are entropy-dense in $\mathcal{M}(X_k)$. Then the ergodic measure $\mathcal{M}^e(X)$ are entropy-dense in $\mathcal{M}(X)$.

The assumptions can be changed using the following observations:

Proposition

For subshifts X, Y,

$$\bar{d}(\mathcal{M}(X), \mathcal{M}(Y)) = \bar{d}(\mathcal{M}^e(X), \mathcal{M}^e(Y)) \leq \underline{d}(X, Y).$$

$$\underline{d}(x, y) = \limsup_{n\to\infty} \frac{1}{n} \sum_{0\leq i<n} |x_i - y_i| \qquad x, y \in \{0,1\}^{\mathbb{N}},$$

$$\underline{d}(X, Y) = \max(\sup_{x\in X} \inf_{y\in Y} \underline{d}(x, y), \sup_{y\in Y} \inf_{x\in X} \underline{d}(x, y)), \quad X, Y \subset \{0,1\}^{\mathbb{N}}.$$

Lemma

For a finite $\mathcal{B} \subset \mathbb{N}$, $\tilde{X}_{\mathcal{B}}$ is a transitive sofic shift, therefore its ergodic measures are entropy-dense in $\mathcal{M}(\tilde{X}_{\mathcal{B}})$.

Lemma

Let $\mathcal{B}$, $\mathcal{B}_k \subset \mathbb{N}$, $\mathcal{B}_k$ be finite for every $k \in \mathbb{N}$ and $\mathcal{B} = \cup_k \mathcal{B}_k$. Then $\underline{d}(\tilde{X}_{\mathcal{B}_k}, \tilde{X}_{\mathcal{B}})$ goes to zero.

We use here Erdos-Davenport's result on $\underline{d}$ and the logarithmic density of integer sets.

To extend the results to $\tilde{X}^*_{\mathcal{B}}$, we use the fact that

$$\underline{d}(\tilde{X}^*_{\mathcal{B}}, \tilde{X}_{\mathcal{B}_k}) \leq \underline{d}(\tilde{X}_{\mathcal{B}}, \tilde{X}_{\mathcal{B}_k}).$$

Other scheme for d-metric

Theorem (decreasing scheme)

Suppose that X_k, $k \in \mathbb{N}$, be a sequence of decreasing subshifts such that $\sum k \in \mathbb{N} \bar{d}(\mathcal{M}^e(X_k), \mathcal{M}^e(X_{k+1}))$ goes to zero, and for every $k \in \mathbb{N}$, the ergodic measures $\mathcal{M}^e(X_k)$ are entropy-dense in $\mathcal{M}(X_k)$. Put $X = \bigcap_{k \in \mathbb{N}} X_k$. Then the ergodic measure $\mathcal{M}^e(X)$ are entropy-dense in $\mathcal{M}(X)$.