Nonexpansiveness in Z^2 symbolic dynamics

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Happy Birthday!

Lesser Celandine
Ficaria verna

Joint Work with Van Cyr and Bryna Kra

Subshifts

Consider the full shift space in d-dimensions $\mathcal{A}^{\mathbb{Z}^d}$ on a finite alphabet $\mathcal{A}$ and the natural action of $\mathbb{Z}^d$ on it.

Definition

Suppose $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is a closed set that is invariant under the $\mathbb{Z}^d$ action, then $\mathbb{Z}^d$ acting on X is a topological $\mathbb{Z}^d$-subshift. *The automorphism group $\mathrm{Aut}(X, \sigma)$, is the group of all homeomorphisms $\phi\colon X \to X$ that commute with the $\mathbb{Z}^d$-action.*

By the Curtis-Hedlund-Lyndon Theorem any automorphism $\phi\colon X \to X$ is given by a block code. An element $x \in X$ is called a *coloring* of $\mathbb{Z}^d$.

There are contributions to the study of $\mathrm{Aut}(X, \sigma)$ by a number of authors including, but not limited to, M. Boyle, G. A. Hedlund, M. Hochman, K. H. Kim, W. Krieger, D. Lind, M. Nasu, F. W. Roush, D. Rudolph, J. Wagoner.

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Information propagation

Definition (cf. Boyle-Lind)

Given a subshift X we say a set $A \subset \mathbb{R}^d$ codes $B \subset \mathbb{R}^d$ if for every $x, y \in X$ with $x(a) = y(a)$ for all $a \in A \cap \mathbb{Z}^d$ we also have $x(b) = y(b)$ for all $b \in B \cap Z^d$.

This is uninteresting if $d = 1$.

Proposition (Morse-Hedlund)

Suppose X is a one-dimensional shift. If $A \subset \mathbb{Z}$ is an interval and A codes a disjoint set $B \subset \mathbb{Z}$ then X is finite.

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Expansive Subspaces

Suppose $X \subset \mathcal{A}^{\mathbb{Z}^2}$ is a topological $\mathbb{Z}^2$-subshift and H is a half space in $\mathbb{R}^2$ bounded by a line L.

Definition (Expansive)

If H codes all of $\mathbb{R}^2$ we say H is expansive. If ℓ is a ray in L whose orientation agrees with the orientation L inherits from H and H is expansive we say that ℓ is an expansive ray. A ray or halfspace which is not expansive is called nonexpansive.

This is slightly non-standard, cf. Boyle-Lind who define L to be an expansive subspace if **both** the half spaces bounded by L are expansive in our sense. For rational directions this is equivalent to usual expansiveness of the sub subdynamics in that direction.

Boyle-Maass and Cyr-Kra consider one-sided expansiveness.

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The Ledrappier example: $d = 2$.

Definition

Let $X \subset \{0,1\}^{\mathbb{Z}^2}$ be the set of $\{0,1\}$ colorings of $\mathbb{Z}^2$ such that for all $x \in X$ and all $(i,j) \in \mathbb{Z}^2$

$$x(i,j) + x(i+1,j) + x(i,j+1) \equiv 0 \mod (2)$$

Note that any two of $\{x(i,j), x(i+1,j), x(i,j+1)\}$ code the third.
Any horizontal line codes the horizontal line above it, but not the horizontal line below it.
The negative x-axis is (or the half space below it) is expansive, but the upper half space is nonexpansive.
Theme: Information propagation is controlled by the geometry of the nonexpansive spaces.

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Expansiveness: $d = 2$.

Observation

Since a shift X is invariant under the $\mathbb{Z}^2$ action, it follows that for every $v \in \mathbb{Z}^2$, if A codes B then $A + v$ codes $B + v$.

Lemma

If $A \subset H$ codes $b \in \mathbb{Z}^2 \setminus H$ then H codes all of $\mathbb{R}^2$ so H is expansive. In this case there is a finite subset $A_0 \subset A$ such that A_0 codes b.

Corollary (cf. Boyle-Lind)

In the space of one-dimensional rays in $\mathbb{R}^2$, being expansive is an open condition and nonexpansive a closed one.

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The spacetime of an automorphism/endomorphism

Definition

If $\phi \in \mathrm{Aut}(X, \sigma)$ *is an automorphism (i.e.* $\phi \circ \sigma = \sigma \circ \phi$*) its* ϕ*-spacetime* $\mathcal{U} = \mathcal{U}(\phi)$ *is a* $\mathbb{Z}^2$*-system, together with a preferred basis for* $\mathbb{Z}^2$*. The set* $\mathcal{U} = \mathcal{U}(\phi)$ *is the closed subset of* $\Sigma^{\mathbb{Z}^2}$ *defined by* $x \in \mathcal{U}$ *if and only if there is* $x_0 \in X$ *with* $\phi^j(x_0)[i] = x(i, j)$ *for all* $(i, j) \in \mathbb{Z}^2$*. The preferred basis consists of* $(1, 0)$*, the "horizontal" and* $(0, 1)$ *the"vertical."*

Note that $(1, 0)$ acts on $\mathcal{U}$ by shifting on each (horizontal) row and the action of $(0, 1)$ can be thought of as a vertical shift or as applying ϕ to each row.

This concept with different terminology was introduced by Milnor.

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For many questions there is no loss of generality in assuming a $\mathbb{Z}^2$-subshift is the spacetime of an automorphism.

Proposition

If X is a $\mathbb{Z}^2$-subshift which has an expansive one-dimensional subspace L then X is isomorphic to a spacetime of an automorphism with L as "horizontal".

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What is coded by a half line?

Definition

If $\phi \in \mathrm{Aut}(Y, \sigma)$ and $n \geq 0$, we say B is ϕ-coded by $A \subset Y$ provided x, y agreeing on A implies $\phi(x), \phi(y)$ agree on B. Let $W^+(n, \phi)$ be the smallest element of $\mathbb{Z}$ such that the ray $[W^+(n, \phi), \infty)$ is ϕ^n-coded by $[0, \infty)$. $W^-(n, \phi)$ is defined similarly.

In other words if x and y agree on $[0, \infty)$, then $\phi^n(x)$ and $\phi^n(y)$ agree on $[W^+(n, \phi), \infty)$ and this is the largest ray with that property.

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Definition (Light Cone)

The future light cone $\mathcal{C}_f(\phi)$ of ϕ is defined to be

$$\mathcal{C}_f(\phi) = \{(i,j) \in \mathbb{Z}^2 \colon W^-(j,\phi) \le i \le W^+(j,\phi),\ j \ge 0\}$$

The past light cone $\mathcal{C}_p(\phi)$ of ϕ is defined to be $\mathcal{C}_p(\phi) = -\mathcal{C}_f(\phi)$. The full light cone $\mathcal{C}(\phi)$ is defined to be $\mathcal{C}_f(\phi) \cup C_p(\phi)$.

The rationale for this terminology is that if $x \in X$ and $j > 0$, then a change in the value of $x(0)$ (with no other changes) can only cause a change in $\phi^j(x)[i]$ if (i,j) lies in the future light cone of ϕ. Similarly a change in the value of $\phi^{-j}(x)[i]$ can only affect $x[0]$ if (i,j) lies in the past light cone of ϕ.

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The asymptotic light cone

The integers $W^+(n)$ and $W^-(n)$ are subadditive so Fekete's Lemma implies the limits $\lim_{n\to\infty} \frac{W^+(n)}{n}$ and $\lim_{n\to\infty} \frac{W^-(n)}{n}$ exist.

Definition

We define

$$\alpha^+(\phi) := \lim_{n\to\infty} \frac{W^+(n)}{n}$$

and

$$\alpha^-(\phi) := \lim_{n\to\infty} \frac{W^-(n)}{n}.$$

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Definition

The ***asymptotic light cone*** *of* ϕ *is defined to be*

$$\mathbb{A}(\phi) = \{(u, v) \in \mathbb{R}^2 \colon \alpha^-(\phi)v \leq u \leq \alpha^+(\phi)v\}.$$

Question

Does there exist a subshift of finite type (X, σ) *and an automorphism* $\phi \in \mathrm{Aut}(X)$ *such that an edge of the asymptotic light cone of* ϕ *has irrational slope? If so, what set of slopes is achievable?*

There are only countably many shifts of finite type, and countably many automorphisms for each, so there are only countably many asymptotic light cones. In particular, all irrational slopes cannot be realized.

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Asymptotic light cone edges

Recall the *theme:* Information propagation is controlled by the geometry of the nonexpansive spaces.

Theorem (Cyr, F., Kra)

The edges of the asymptotic light cone are nonexpansive subspaces.

Conversely, if L is a subspace in the frontier of the expansive subspaces it is (after change of co-ordinates and recoding), the edge of the asymptotic light cone of an automorphism.

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Let $A(\phi)$ be the length of the smallest interval containing $0, \alpha^-(\phi)$ and $\alpha^+(\phi)$. It is called the asymptotic spread.

Theorem (V. Cyr, J. F., B. Kra)

If $\phi \in \mathrm{Aut}(X)$, then the topological entropies $h(\phi)$ and $h(\sigma)$, satisfy

$$h(\phi) \leq A(\phi)h(\sigma),$$

An earlier similar result for automorphisms of the full shift preserving the uniform measure was proved by P. Tisseur

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What groups can be subgroups of $\mathrm{Aut}(X, \sigma)$?

Proposition (Cyr, F, Kra, Petite)

The following groups cannot be isomorphic to subgroups of $\mathrm{Aut}(X, \sigma)$, if $h(\sigma) = 0$.

- $\mathrm{SL}(k, \mathbb{Z})$ *for any* $k \geq 3$
- $\mathrm{SL}(2, \mathbb{Z}[1/p])$, *for any prime* p
- *The Baumslag-Solitar group* $G = \langle a, b \colon bab^{-1} = a^n \rangle$.

Polygonal subshifts (joint with Kra)

Suppose X is a $\mathbb{Z}^2$-subshift, P is a convex polygon with vertices in $\mathbb{Z}^2$, and v is a vertex of P. If $P \setminus \{v\}$ X-codes $\{v\}$, then we say that $\mathcal{P}$ is a *coding polygon* for the vertex v. Note that the assumption that $P \setminus \{v\}$ codes $\{v\}$ implies that $(P+u) \setminus \{v+u\}$ X-codes $\{v+u\}$ for all $u \in \mathbb{Z}^2$.)

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Polygonal examples

- The Ledrappier example $\mathcal{L}$ is polygonal with respect to $\mathcal{P}$, the triangle with vertices $\{(0,0),(1,0),(0,1)\}$. Also a non-abelian analog with $\mathcal{A}$ any finite group.
- Einsiedler has shown there are closed $\mathbb{Z}^2$-invariant subsystems of $\mathcal{L}$ realizing all horizontal directional entropies in $[0, \ln(2)]$.
- Suppose X is polygonal with respect to $\mathcal{P}$ it is polygonal with respect to $n\mathcal{P}$ for $n \geq 1$.
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- The Ledrappier example $\mathcal{L}$ is polygonal with respect to $\mathcal{P}$, the triangle with vertices $\{(0,0),(1,0),(0,1)\}$. Also a non-abelian analog with $\mathcal{A}$ any finite group.
- Einsiedler has shown there are closed $\mathbb{Z}^2$-invariant subsystems of $\mathcal{L}$ realizing all horizontal directional entropies in $[0, \ln(2)]$.
- Suppose X is polygonal with respect to $\mathcal{P}$ it is polygonal with respect to $n\mathcal{P}$ for $n \geq 1$.
- The Cartesian product of two polygonal shifts with different polygons.

Observation

If X is polygonal with respect to $\mathcal{P}$ then every subspace which is not parallel to a side of $\mathcal{P}$ is expansive. Moreover if $\mathcal{P}$ has no two parallel edges then every (positively oriented) edge of $\mathcal{P}$ determines a positively expansive ray for X of $\mathcal{P}$.

Closing rays

Definition

Suppose H is an open nonexpansive half space with boundary L and $\ell \subset L$ is the corresponding nonexpansive ray. We say H and ℓ are closing *if ℓ is rational and there is an $N > 0$ such any block B of length N in $L \cap \mathbb{Z}^2$ has the property that $B \cup H$ codes L.*

Example: The positive x-axis in the Ledrappier example. The name comes from the fact that for S a spacetime of an endomorphism ϕ, the upper half space (and positive x-axis) are closing if the endomorphism ϕ is right and left closing in the sense of symbolic dynamics.

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Characterizing polygonal systems

Suppose X is a $\mathbb{Z}^2$ subshift. with only finitely many nonexpansive rays. We call this the *nonexpansive ray set* for X.

Theorem (J. F., B. Kra)

Suppose X is a $\mathbb{Z}^2$ subshift. Then X is isomorphic to a polygonal shift iff X has only finitely many nonexpansive subspaces and they are rational and closing.

In particular the the geometry of the nonexpansive rays determines the geometry of the the polygon, e.g. the number of sides and the angles they make with the axes.

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THANK YOU!

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Directional entropy

Definition

If X is a $\mathbb{Z}^2$-subshift and $v \in \mathbb{Z}^2$ then the directional entropy *$h_v(x)$ in direction v is the topological entropy h_v of T_v the element of the $\mathbb{Z}^2$ action on X determined by v. It can be extended by homogeneity to all of $\mathbb{Q}^2$ and by continuity to all of $\mathbb{R}^2$. The* entropy semi-norm *on $\mathbb{R}^2$ is defined by $\|v\|_X = h_v(X)$.*

Proposition (Milnor)

If X is a $\mathbb{Z}^2$-subshift which is polygonal then $\|v\|_X$ is either identically 0 or a norm. If X, Y are triangular with the same triangle and non-trivial semi-norms then there is a constant c such that $\|v\|_X = c\|v\|_Y$.

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Directional entropy

Proposition (J. F., B. Kra)

Suppose X is a polygonal $\mathbb{Z}^2$*-subshift with rational polygon* $\mathcal{P}$*. has no parallel sides. Then if* $\mathcal{F}(\mathcal{P})$ *is the family of all* $\mathbb{Z}^2$*-subshifts which are polygonal with respect to* $\mathcal{P}$ *and which have nontrivial entropy norms, then* $\mathcal{F}(\mathcal{P})$ *is a* quasi-conformal family*, i.e. there is a uniform dilatation constant* $D > 0$*, depending only on* $\mathcal{P}$*, which has the property that for any* $u, v \in S^1$ *we have*

$$\frac{1}{D} \leq \frac{h_u(X)}{h_v(X)} \leq D.$$

for all $X \in \mathcal{F}(\mathcal{P})$*.*