Folding points and endpoints of chainable continua

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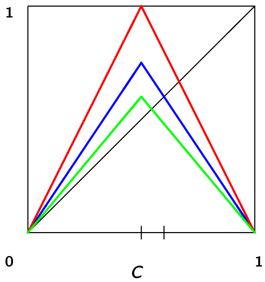
Joint work with Lori Alvin (Furman University), Ana Anušić (USP) and Henk Bruin (University of Vienna)

Figure: Hemerocallis lilioasphodelus (rumena maslenica)

Introduction

Let $I := [0,1]$ and let

The *tent map family* $T_s(x) := \begin{cases} sx, x \in [0,1/2] \\ s(1-x), x \in [1/2,1] \end{cases}$



for $s \in (0,2]$, where c denotes the *critical point* of T_s.

Inverse limit spaces

We define the *inverse limit space* $X_s = \varprojlim(T_s, I)$ with a single bonding map T_s by

$$X_s := \{x := (x_0, x_1, x_2, x_3, \dots) \in I^\infty;\, T_s(x_{n-1}) = x_n, \forall\, n \in \mathbb{N}\},$$

equipped with a metric

$$d(x,y) := \sum_{i \geq 0} \frac{|x_i - y_i|}{2^i}$$

for every $x, y \in X_s$.
and the *shift homeomorphism* $\sigma : X_s \to X_s$, defined by

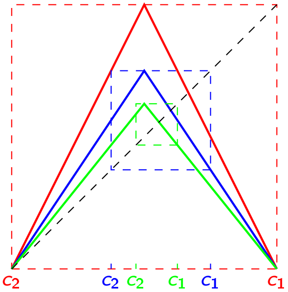
$$\sigma((x_0, x_1, x_2, \dots)) := (T_s(x_0), T_s(x_1), T_s(x_2), \dots)$$

and *coordinate projections* $\pi_n : X_s \to I$ by $\pi_n(x) := x_n \;\; \forall n \in \mathbb{N}_0$.

Tent inverse limit spaces

A space is called a *continuum* if it is a compact connected metric space. Denote by $[c_2, c_1]$ the *core* of T_s, where $c_k := T^k(c)$.

$\longrightarrow$ X_s *chainable* continua ($\forall \epsilon > 0$ $\exists \epsilon$-mapping from X_s to I and thus X_s planar and $X_s = \mathcal{C} \cup X'_s = \varprojlim([c_2, c_1], T_s)$ $\forall s \in (\sqrt{2}, 2)$ and $\mathcal{C}$ is a ray (Bing, Bennet (1960's)).



How to think about spaces X_s?

$\longrightarrow$ Fat maps

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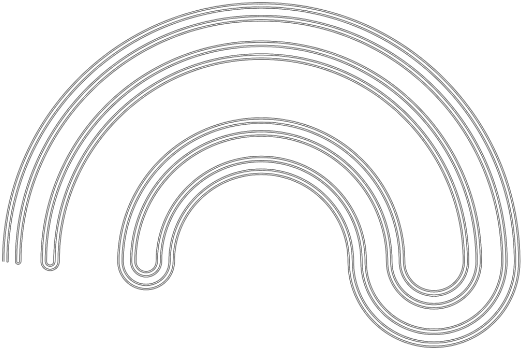

Figure: Continuum X_2

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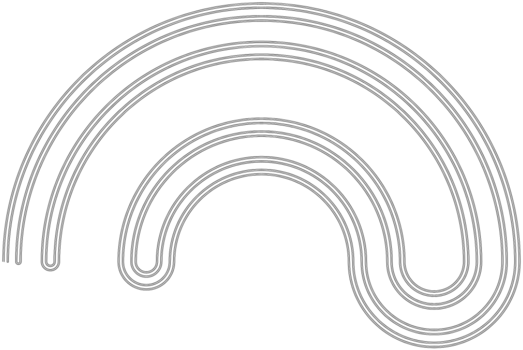

Figure: Continuum X_2

$\longrightarrow$ Thm(Barge, Bruin,Štimac, 2013): $\forall s \neq s' \in (\sqrt{2}, 2]$, then $X_s \not\approx X_{s'}$.

Why are X_s interesting to study?

- X_s are for some s homeomorphic to global attractors of planar diffeomorphisms (Hénon maps $H_{a,b}(x,y) \longmapsto (1 - ax^2 + y, bx)$ for some $a \in [1,2]$ and b small - Barge & Holte (1995))
- X_s as a global attracting sets Λ of a plane homeomorphism H such that $H|_\Lambda$ and σ are topologically conjugate (*e.g.* Barge & Martin, (1990), Boyland, de Carvalho & Hall (2012))
- simplest parametrized family (containing folding points) plane non-separating one-dimensional attractors.

Folding points and endpoints of a chainable continuum K

Folding points $\mathcal{F} \subset K$: points with a neigb. $\not\approx C \times (0,1)$ where C is a totally disconnected set.

Endpoints $\mathcal{E} \subset K$: points $x \in K$ s.t. $\forall A, B \subset K$ subcontinua containing x it hold that $A \subset B$ or $B \subset A$.

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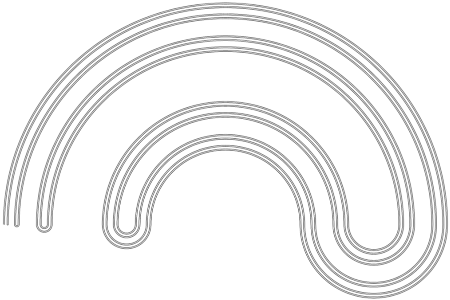

Figure: Knaster continuum X_2

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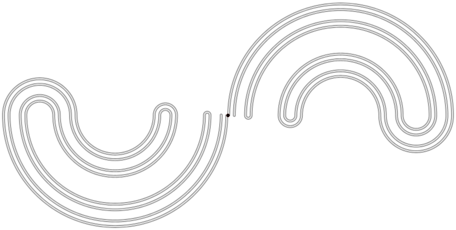

Figure: Continuum $X_{\sqrt{2}}$

What is $\mathcal{F} \subset X_s$ depending on s?

$X_2 \ldots \exists\ x \in \mathcal{E}$

$X_{\sqrt{2}} \ldots \exists\ x \in \mathcal{F} \setminus \mathcal{E}$

$\longrightarrow$Thm (Barge, Martin (1994)): Say $s \in (\sqrt{2}, 2)$. If c is (pre)periodic with (pre)period n then X_s' has n points in $\mathcal{E}$ ($\mathcal{F} \setminus \mathcal{E}$).

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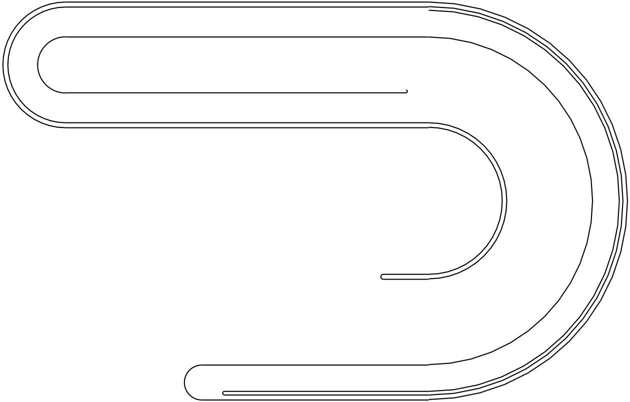
$\longrightarrow$Thm (Barge, Brucks, Diamond (1996)): For a dense G_δ set of parameters from $[\sqrt{2}, 2]$ it holds that every open set in X_s' contains a homeomorphic copy of every tent inverse limit space (also with varying bonding maps).

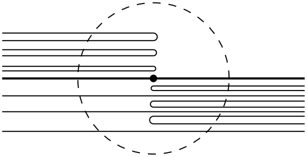
Representation of X_s

$A \subset X_s$ is *basic arc* if $\pi_0(A)$ maximal injective s.t $x \in A \subset X$.

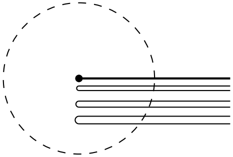


Folding points $\mathcal{F} \setminus \mathcal{E}$ and endpoints $\mathcal{E} = \mathcal{E}_F \cup \mathcal{E}_S \cup \mathcal{E}_N$

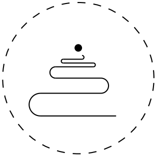
$\mathcal{F} \setminus \mathcal{E}$ -non-end folding point



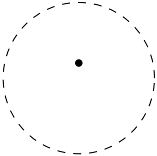
$\mathcal{E}_F$ -flat endpoint



$\mathcal{E}_S$ -spiral endpoint



$\mathcal{E}_N$ -nasty endpoint



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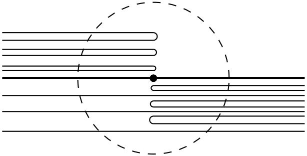
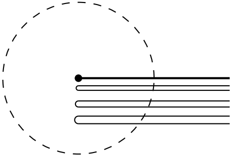
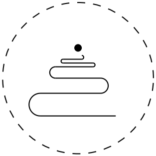


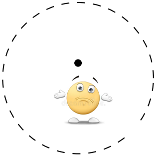
$\mathcal{E}_F$ -flat endpoint



$\mathcal{E}_S$ -spiral endpoint



$\mathcal{E}_N$ -nasty endpoint



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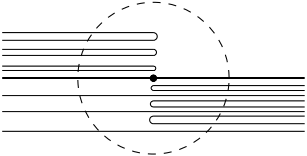
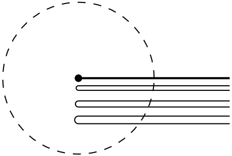
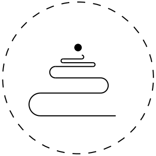


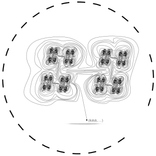
$\mathcal{E}_F$ -flat endpoint



$\mathcal{E}_S$ -spiral endpoint



$\mathcal{E}_N$ -nasty endpoint



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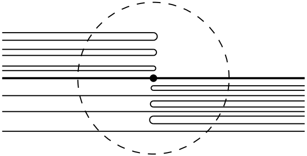
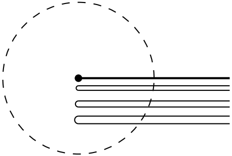
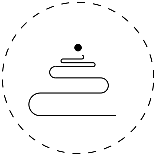


$\mathcal{E}_F$ -flat endpoint



$\mathcal{E}_S$ -spiral endpoint



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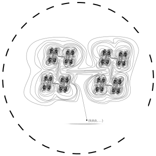

Thm (Alvin, Anušić, Bruin, Č., 2019): Say that $\mathrm{orb}(c)$ is infinite. Then, the sets $\mathcal{F} \setminus \mathcal{E}$, $\mathcal{E}_F$, $\mathcal{E}_S$, $\mathcal{E}_N$ are dense, whenever non-empty.

Open questions

Q1: Let c recurrent and $\mathrm{Orb}(c)$ infinite. Are $\mathcal{E}_S$, $\mathcal{E}_N$, $\mathcal{E}_F$ uncountable when non-empty?

Q2: (Boyland, de Carvalho, Hall, 2017) Say $\omega(c) = [c_2, c_1]$. Which of $\mathcal{E}_S, \mathcal{E}_N$ is topologically typical?

Q3: Say $\omega(c) \neq [c_2, c_1]$. Can there exist $x \in \mathcal{E}_N$?

When $\mathcal{F} = \mathcal{E}$ in X_s'?

$\longrightarrow$ Let $s \in (\sqrt{2}, 2)$. Then $\exists x \in \mathcal{E} \subset X_s' \iff c$ is recurrent. (Barge, Martin 1994)

$\longrightarrow$ $x \in \mathcal{F} \subset X_s' \iff \forall k \in \mathbb{N}\ \pi_k(x) \in \omega(c)$ (Raines 2006)

$\longrightarrow$ If $Q(k) \to \infty$ and $T_s|_{\omega(c)}$ bijective $\implies \mathcal{F} = \mathcal{E}$ but $\not\Leftarrow$ (Alvin & Brucks 2011, Alvin 2013)

Persistent recurrence

Def: Let $x = (x_0, x_1, \dots) \in X'_s$ and let $J \subset I$ be an interval. The sequence $(J_n)_{n \in \mathbb{N}_0}$ of intervals is called a *pull-back* of J along x if $J = J_0$, $x_k \in J_k$ and J_{k+1} is the largest interval such that $T_s(J_{k+1}) \subset J_k$ for all $k \in \mathbb{N}_0$. A pull-back is *monotone* if $c \notin \mathrm{Int}(J_n)$ for every $n \in \mathbb{N}$.

Def (Blokh, Lyubich (1991)): Let c be recurrent. The critical point c is *reluctantly recurrent* if there is $\varepsilon > 0$ and an arbitrary long (but finite!) backward orbit $\bar{y} = (y, y_1, \dots, y_l)$ in $\omega(c)$ such that the ε-neighbourhood of $y \in I$ has monotone pull-back along $\bar{y}$. Otherwise, c is *persistently recurrent*.

Result

Lemma: Let $y \in \omega(c)$, $y \in \mathrm{Int}(J)$ where $J \subset I$ and assume that for every $i \in \mathbb{N}$ the set J can be monotonically pulled-back along $c_{n_i+1}, \ldots, c_1$, where $J \ni c_{n_i+1} \neq y$. Then J can be monotonically pulled-back along some infinite backward orbit $y, y_1, y_2, \ldots$, where $y_i \in \omega(c)$ for every $i \in \mathbb{N}$.

Theorem: (Alvin, Anušić, Bruin, Č., 2019) $\mathcal{F} = \mathcal{E} \subset X_s' \iff c$ is persistently recurrent.

Problems with generalisation on chainable continua K

Def: A point $x \in K$ is a *Lelek endpoint* if it is endpoint of every arc containing K.

$\longrightarrow$ Let $x \in X_s'$. Point x is a Lelek endpoint $\iff$ x is a standard endpoint.

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Monotone pullbacks possible if e.g. there are no double spirals as subcontinua in the continuum K .

Thank you!