Entropy on modules over the group ring of a sofic group

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What's the name of this flower?

Blue lotus/Nymphaea caerulea/ 蓝莲花（lan lianhua）

“Live in the silt but not imbrued”

Overview

Peters' Bridge theorem states that the topological entropy of a compact abelian group automorphism can be formulated in terms of its dual action. The proof works for an amenable group action.

Since the notion of topological entropy has been extended to sofic group actions, it is natural to ask whether Bridge theorem can be extended to such a case. This is our main concentration.

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Example

$\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$; $\widehat{\mathbb{Z}} \cong \mathbb{R}/\mathbb{Z}$; and $\widehat{\oplus_I \mathbb{Z}} \cong (\mathbb{R}/\mathbb{Z})^I$.

Group rings

When A carries a group action of a countable group Γ by automorphisms, A becomes a $\mathbb{Z}\Gamma$-module by linear extension.

Definition

The **group ring $\mathbb{Z}\Gamma$ of Γ with coefficients in $\mathbb{Z}$** consists of all finitely supported functions $f = \sum_{s\in\Gamma} f_s s$ on Γ. The algebraic operations on $\mathbb{Z}\Gamma$ are defined by

- $\sum_{s\in\Gamma} f_s s + \sum_{s\in\Gamma} g_s s = \sum_{s\in\Gamma}(f_s + g_s)s$;
- $\big(\sum_{s\in\Gamma} f_s s\big)\big(\sum_{t\in\Gamma} g_t t\big) = \sum_{s,t\in\Gamma} f_s g_t (st)$.

Algebraic actions

Moreover, $\widehat{A}$ will carry a Γ-action by continuous automorphisms from the group ring module structure of A, i.e.

$$\langle s\chi, a\rangle := \langle \chi, s^{-1}a\rangle$$

for all $\chi \in \widehat{A}, a \in A$, and $s \in \Gamma$. We call such an action $\Gamma \curvearrowright \widehat{A}$ an **algebraic action**.

Example

For any $f \in \mathbb{Z}\Gamma$, $\widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f} = \{x \in (\mathbb{R}/\mathbb{Z})^\Gamma : fx = 0 \text{ in } (\mathbb{R}/\mathbb{Z})^\Gamma\}$. In particular, let $f = 2\delta_{e_\Gamma}$, then $\widehat{\mathbb{Z}\Gamma/2\mathbb{Z}\Gamma} = (\mathbb{Z}/2\mathbb{Z})^\Gamma$ is the Bernoulli shift.

Peters' formula

Theorem (Peters, 1979 [6])

If Γ is an amenable group and $\Gamma \curvearrowright X$ is an algebraic action, then the topological entropy of $\Gamma \curvearrowright X$ coincides with the algebraic entropy of $\widehat{X}$. That is,

$$h(\Gamma \curvearrowright X) = sup_{E \subseteq \widehat{X}} \lim_{F} \frac{|\sum_{s \in F} s^{-1} E|}{|F|},$$

where E ranges over all finite subsets of $\widehat{X}$ and the limit $\lim_F$ is taken along Følner sequences.

Strategy: passes to measure-theoretic entropy with respect to Harr meausre and applies Fourier Inverse Theorem.

Beyond amenable group actions

Since entropy theory has been extended to more general group action case, we may consider extending Peters' Bridge theorem correspondingly.

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- Li-Liang (2015) [5]: one can extend the notion of algebraic entropy to the case of sofic group actions.

Sofic groups

The class of sofic groups was introduced by Gromov in 1999 when he dealt with Gottschalk's conjecture. This class unifies the amenable groups and residually finite groups. Many groups remain to be shown sofic. However, there is no nonsofic groups found yet.

Example

The free group F_2 with two generators is sofic since it admits a sequence of asymptotically faithful actions on finite sets:

$$\{F_2 \curvearrowright (\mathbb{Z}/n\mathbb{Z})^2\}_{n=1}^{\infty}.$$

Sofic groups

Definition

A countable discrete group Γ is **sofic** if it admits an approximation sequence of maps $\Sigma := \{\sigma_i : \Gamma \to \mathrm{Sym}([d_i])\}_{i=1}^{\infty}$ satisfying that

1. for any $s \neq t \in \Gamma$, $|\{v \in [d_i] : \sigma_i(st)(v) = \sigma_i(s)\sigma_i(t)(v)\}|/d_i \to 1$;
2. for any $s \neq e_\Gamma \in \Gamma$, $|\{v \in [d_i] : \sigma_i(s)(v) \neq v\}|/d_i \to 1$;
3. $d_i \to \infty$.

Idea of defining sofic topological entropy

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Idea of defining sofic topological entropy

Let Γ act on a compact metric space (X, ρ) by homeomorphisms.

Idea: use the "pseudo-actions" $\{\Gamma \curvearrowright^{\text{``}\sigma_i\text{''}} [d_i]\}_i$ to model the real actions $\Gamma \curvearrowright X$. That is, we consider those $\varphi : [d_i] \to X$ such that for "many" $s \in \Gamma$ the diagram

$$\begin{array}{ccc} [d_i] & \xrightarrow{\sigma_i(s)} & [d_i] \\ \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle\varphi} \\ X & \xrightarrow{s} & X \end{array}$$

"approximately commutes".

Sofic topological entropy

Let $\delta > 0$ and $F \subseteq \Gamma$ a finite subset. Consider the subspace of X^{d_i}

$$\mathrm{Map}(\rho, F, \delta, \sigma_i) := \{\varphi : [d_i] \to X : \rho_2(s\varphi, \varphi \circ \sigma_i(s)) \leq \delta \ \forall\ s \in F\}.$$

Definition

The **sofic topological entropy** of $\Gamma \curvearrowright X$ is

$$h_\Sigma(\Gamma \curvearrowright X) := \sup_{\varepsilon>0} \inf_{F\subseteq\Gamma} \inf_{\delta>0} \overline{\lim_{i\to\infty}} \frac{\log |N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma_i), \rho_\infty)|}{d_i},$$

where $N_\varepsilon(\cdot, \rho_\infty)$ denotes the cardinality of a maximal $(\varepsilon, \rho_\infty)$-separating subset.

Example

$h_\Sigma(\Gamma \curvearrowright \{0,1\}^\Gamma) = \log 2.$

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That is, consider $\psi : [d_i] \to \mathcal{M}$. Then

$$s\psi \approx \psi \circ \sigma_i(s)$$
$$\Updownarrow$$
$$\widehat{s\psi} \approx \widehat{\psi \circ \sigma_i(s)}$$
$$\Updownarrow$$
$$< \widehat{\psi}(v), b >=< \widehat{\psi}(\sigma_i(s)v), sb > \ \textit{for all } v \in [d_i] \textit{ and "many" } b \in \mathcal{M}.$$

Sofic algebraic entropy

Let $B \subseteq \mathcal{M}$ and $F \subseteq \Gamma$ be finite subsets. Denote by $\mathscr{M}(B, F, \sigma_i)$ the abelian subgroup of $\mathcal{M}^{\otimes d_i}$ generated by

$$\delta_v \otimes b - \delta_{\sigma_i(s)v} \otimes sb$$

for all $v \in [d_i], b \in B$, and $s \in F$.

Definition

The **sofic algebraic entropy** of $\mathcal{M}$ is

$$h_\Sigma(\mathcal{M}) := \sup_{A \subseteq \mathcal{M}} \inf_{F \subseteq \Gamma} \inf_{B \subseteq \mathcal{M}} \overline{\lim_{i \to \infty}} \frac{\log |A^{d_i} / \mathscr{M}(B, F, \sigma_i)|}{d_i},$$

where A ranges over all finite subsets of $\mathcal{M}$.

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Example

$h_\Sigma((\mathbb{Z}/2)\Gamma) = \log 2.$

Main result

Theorem (Liang, 2019 [4])

Let $\mathbb{F}$ *be a finite field and* $\mathcal{M}$ *a finitely generated* $\mathbb{F}\Gamma$*-module. Then*

$$h_\Sigma(\mathcal{M}) = h_\Sigma(\widehat{\mathcal{M}}).$$

Main result

Theorem (Liang, 2019 [4])

Let $\mathbb{F}$ be a finite field and $\mathcal{M}$ a finitely generated $\mathbb{F}\Gamma$-module. Then

$$h_\Sigma(\mathcal{M}) = h_\Sigma(\widehat{\mathcal{M}}).$$

Sketch of proof:
Step 1. Use modified addition formula to pass to finitely presented case;
Step 2. Establish the equality by combining an approximation formula of sofic algebraic entropy and an approximation formula of sofic topological entropy given by Gaboriau-Seward [2].

References

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Thank you for your attention!