Zigzags in interval inverse limits

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Heliconia rostrata (false bird of paradise or lobster claw)

Inverse limits on intervals

Let $I = [0,1]$ be the unit interval.

For $i \in \mathbb{N}$ let $f_i \colon I \to I$ be continuous (surjection).

$$I \xleftarrow{f_1} I \xleftarrow{f_2} I \xleftarrow{f_3} I \xleftarrow{f_4} I \xleftarrow{f_5} I \ldots$$

The inverse limit space of the inverse system $\{I, f_i\}_{i \in \mathbb{N}}$ **is:**

$$X = \varprojlim\{I, f_i\} = \{(x_0, x_1, x_2, x_3, \ldots) : f_i(x_i) = x_{i-1}\} \subset I^\infty,$$

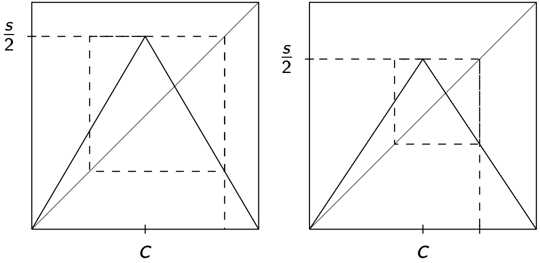
with the product topology.

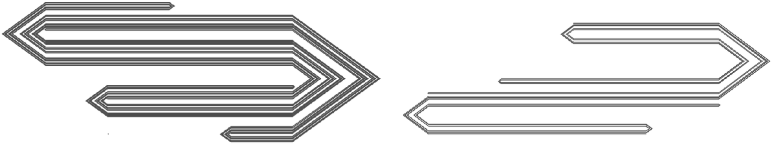
Coordinate projections $\pi_i \colon X \to I$, $\pi_i((x_0, x_1, x_2, \ldots)) = x_i$.

X is a **continuum** (compact, connected, metric), and **chainable** (admits arbitrary small covers whose nerves are arcs).

Unimodal inverse limits $X_s = \varprojlim\{I, T_s\}$

$T_s : I \to I,\ T_s(x) = \min\{sx, s(1-x)\},\ s \in [0,2].$





Planar embeddings

Theorem (Bing 1951)

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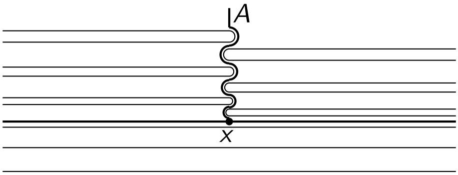
A point $x \in X \subset \mathbb{R}^2$ is called **accessible** if there is an arc $A \subset \mathbb{R}^2$ such that $A \cap X = \{x\}$.



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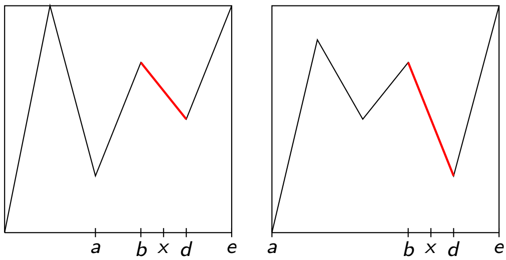
Specially, yes for unimodal inverse limits (A, Bruin, Činč 2016).

Zigzags

Let $f\colon I \to I$ be a continuous piecewise linear surjection. We say that f has a **zigzag** if there exist critical points $a < b < d < e \in I$ of f such that $f|_{[b,d]}$ is one-to-one and either

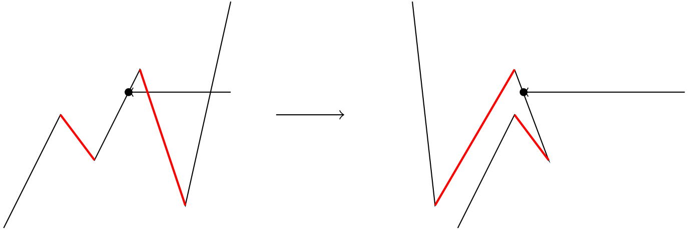
1. $f(b) > f(d)$, $f(a') < f(e)$ for all $a' \in [a, b]$, and $f(e') > f(a)$ for all $e' \in [d, e]$, or
2. $f(b) < f(d)$, $f(a') > f(e)$ for all $a' \in [a, b]$, and $f(e') < f(a)$ for all $e' \in [d, e]$.

We say that $x \in [b, d]$ is **contained in a zigzag of** f.



Zigzags

The idea is that a point x is **not** in a zigzag of f if and only if there exists an arc $\alpha\colon I \to \{(x,y) : x < 0\}$ such that $\pi_y(\alpha(x)) = f(x)$ for all $x \in I$ and $\alpha(x)$ can be accessed by $[\alpha(x), (0, f(x))]$.



Theorem (A., Bruin, Činč 2018)

Let $X = \varprojlim\{I, f_i\}$ where $f_i\colon I \to I$ are continuous surjections. If $x = (x_0, x_1, x_2, \dots) \in X$ is such that x_i is not in a zigzag of f_i for all $i \in \mathbb{N}$, then there exists an embedding of X in the plane such that x is accessible.

Corollaries

Corollary

Let $X = \varprojlim\{I, f_i\}$ *where* $f_i\colon I \to I$ *are continuous surjections which do not have zigzags for all* $i \in \mathbb{N}$. *Then for every* $x \in X$ *there exists an embedding of* X *in the plane such that* x *is accessible.*

Corollary (A, Bruin, Činč 2016)

For every unimodal inverse limit space X *and every* $x \in X$ *there exists an embedding of* X *in the plane such that* x *is accessible.*

Two definitions of an endpoint

Let X be a one-dimensional continuum.

Definition 1

We say that $x \in X$ is an endpoint of X if $x \in A \cap B$ implies $A \subset B$ or $B \subset A$ for every subcontinua $A, B \subset X$.

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Definition 2 (Lelek 60s)

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Question

For which one-dimensional X are the two definitions equivalent?
For which chainable continua X?

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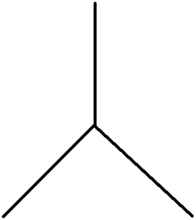

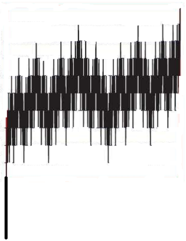

Let $X = \varprojlim\{I, f_i\}$ and $x = (x_0, x_1, \dots) \in X$. For $i \in \mathbb{N}_0$ we define i-**basic arc** $A_i(x)$ as maximal arc in X such that $x \in A_i(x)$ and $\pi_i|_{A_i(x)}$ is one-to-one (can be degenerate).

Endpoints in unimodal inverse limits

Let $X = \varprojlim\{I, f_i\}$ and $x = (x_0, x_1, \dots) \in X$. For $i \in \mathbb{N}_0$ we define i-**basic arc** $A_i(x)$ as maximal arc in X such that $x \in A_i(x)$ and $\pi_i|_{A_i(x)}$ is one-to-one (can be degenerate).

Theorem (Bruin 1999)

Let $X = \varprojlim\{I, T_s\}$ and $x \in X$. Then x is an endpoint of X if and only if x is an endpoint of $A_i(x)$ for every $i \in \mathbb{N}_0$.

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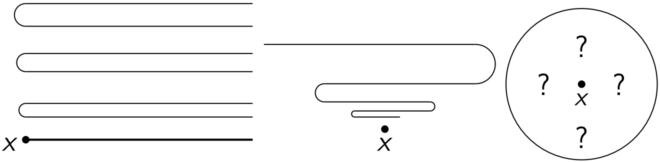

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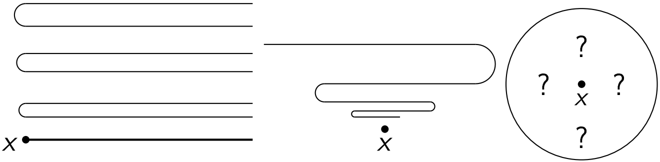

For unimodal inverse limits, the two definitions of endpoints are equivalent.

The third definition of an endpoint

Definition 3

We say that $x \in X = \varprojlim\{I, f_i\}$ is a B-endpoint if it is an endpoint of $A_i(x)$ for every $i \in \mathbb{N}_0$.

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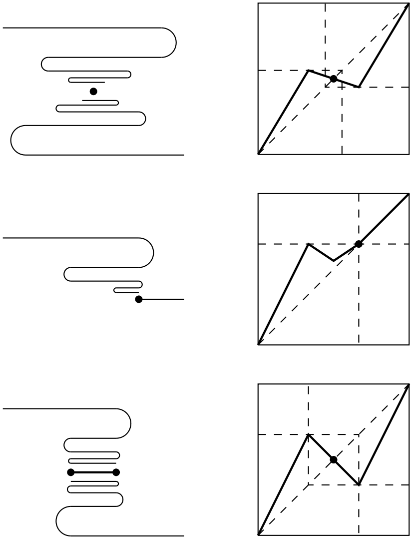
Definition 2 implies Definition 3.

B-endpoints which are not endpoints



Endpoints in zigzag-free interval inverse limits

Theorem

Let $X = \varprojlim\{I, f_i\}$ *and assume that every* f_i *is zigzag-free. Then* $x \in X$ *is an endpoint if and only if it is a B-endpoint. Specially, all three definitions of endpoints are equivalent.*

Theorem

Let $X = \varprojlim\{I, f_i\}$ and assume that every f_i is zigzag-free. Then $x \in X$ is an endpoint if and only if it is a B-endpoint. Specially, all three definitions of endpoints are equivalent.

Sketch of proof:

Assume $x = (x_0, x_1, \ldots)$ is not an endpoint, so there are subcontinua $A, B \subset X$ such that $x \in A \cap B$ and $A \setminus B, B \setminus A \neq \emptyset$. Let $A_i = \pi_i(A), B_i = \pi_i(B)$, $i \in \mathbb{N}_0$ be coordinate projections. They are all intervals, $x_i \in A_i \cap B_i$ for every i, and there exists $N \in \mathbb{N}$ such that $A_i \setminus B_i, B_i \setminus A_i \neq \emptyset$, for all $i > N$.

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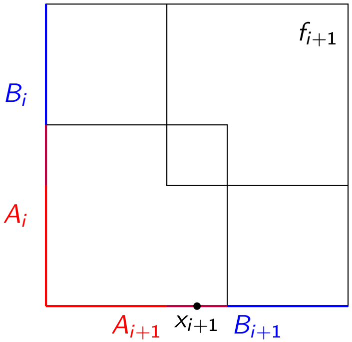
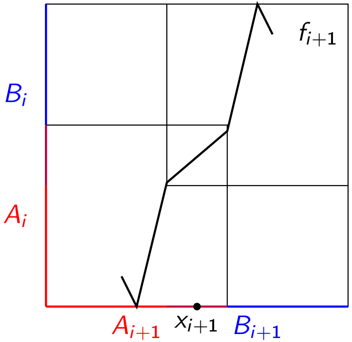


Since f_{i+1} does not contain a zigzag, there exists an interval $(l_{i+1}, r_{i+1}) \ni x_{i+1}$ such that $f_{i+1}|_{[l_{i+1}, r_{i+1}]} : [l_{i+1}, r_{i+1}] \to A_i \cup B_i$ is one-to-one and surjective.



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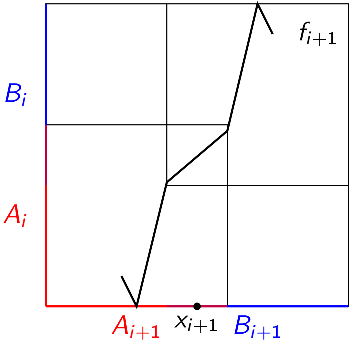
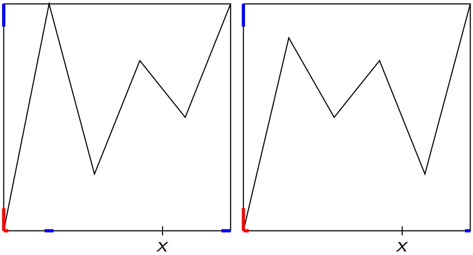


So we can find an arc $A := [a_N, b_N] \xleftarrow{f_{N+1}} [a_{N+1}, b_{N+1}] \xleftarrow{f_{N+2}} [a_{N+2}, b_{N+2}] \xleftarrow{f_{N+3}} [a_{N+4}, b_{N+4}] \xleftarrow{f_{N+4}} \dots$ such that $x \in Int(A) \subset Int(A_N(x))$, so x is not a B-endpoint.

Barge-Martin characterization of endpoints

Theorem (Barge and Martin 1994)

Let $f : I \to I$ be continuous. Then $(x_0, x_1, \dots) \in X = \varprojlim\{I, f\}$ is an endpoint of X if and only if for every $i \in \mathbb{N}$, every interval $J_i = [a_i, b_i] \ni x_i$ and every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that if $J_{i+N} = [a_{i+N}, b_{i+N}]$ is an interval with $x_{i+N} \in J_{i+N}$ and $f^N(J_{i+N}) = J_i$, then x_{i+N} does not separate $f^{-N}([a_i, a_i + \varepsilon])$ from $f^{-N}([b_i - \varepsilon, b_i])$.



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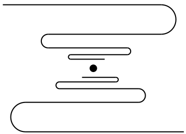

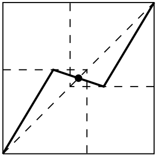

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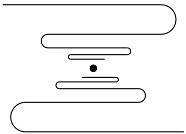

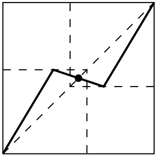

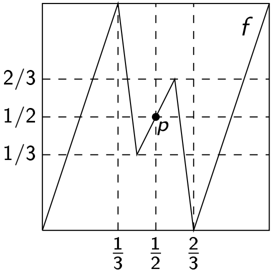

Example by Piotr Minc (2001) suggested as a candidate for a counterexample to the Nadler-Quinn problem.
Map is long-branched and leo, so all basic arcs are sufficiently long, and every proper subcontinuum is an arc.
So point is an endpoint if and only if it is a B-endpoint. Here, only $(0, 0, \dots)$ and $(1, 1, \dots)$ are endpoints.

Questions

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For which $X = \varprojlim\{I, f_i\}$ is it true that every B-endpoint is an endpoint? For which one-dimensional X is every L-endpoint an endpoint? What is we restrict to chainable continua X?

Thank you!

Happy birthday Michał!